Dressing preserving the fundamental group

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Abstract

In this note we consider the relationship between the dressing action and the holonomy representation in the context of constant mean curvature surfaces. We characterize dressing elements that preserve the topology of a surface and discuss dressing by simple factors as a means of adding bubbles to a class of non finite type cylinders.

Introduction. The equation for a harmonic map from a Riemann surface to a Riemannian symmetric space has a zero-curvature representation, and so corresponds to a loop of flat connections. Uhlenbeck discovered in her study [28] of harmonic maps into a compact Lie group G that such maps correspond to certain holomorphic maps into the based loop group of G and used this to define the *dressing action* of a certain loop group on the space of harmonic maps. Dressing, or the *vesture method* [30], was first developed in soliton theory to generate new solutions from old by solving a matrix Riemann problem. Unfortunately, the new solution does not automatically inherit properties of the old, such as domain, periods or asymptotics nor is it easy to control such properties when solving a matrix Riemann problem.

When the target is the two dimensional round sphere, harmonic maps are precisely the Gauss maps of surfaces with constant mean curvature [22]. In the article [10] a method was presented by which all conformally immersed surfaces with constant mean curvature (CMC) can be obtained. The construction involves solving a meromorphic linear differential system with values in a loop group and then Iwasawa decomposing the solution to obtain the extended unitary frame of the surface. Both these steps make it difficult to keep track of the topology. Variation of the initial condition corresponds to the dressing action and is an integral part of the theory. Not only can dressing be used to close periods, but also to generate new CMC surfaces from old ones without altering the topology. In this work, we look into the relationship between dressing and topology in the framework of meromorphic ODE's and loop group factorizations in the context of CMC surfaces.

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The question, when a conformal CMC immersion factors through a given surface has been discussed for tori [1], [6], [21], cylinders [9], [13], [16] and trinoids [2], [11], [16], [25]. It will be interesting to see more results in this direction, with particular emphasis on complete, non-compact surfaces with genus and compact surfaces of genus $g \geq 2$.

Sterling & Wente [26] discovered that one can add "bubbles" to a round cylinder while preserving the mean curvature by Bianchi– Bäcklund transformations. In [13] it was shown that these "bubbletons" can be produced by dressing from the standard round cylinder by the so called *simple factors* of Terng and Uhlenbeck [27]. Subsequently, it was shown in [18] that the simple bubbletons in [13] coincide with those of Sterling and Wente [26]. Recently, A. Mahler [19] has shown that indeed Bianchi – Bäcklund transformations can be achieved by dressing. Further, it was recently shown in [17] that it is also possible to dress CMC surfaces homeomorphic to the n-punctured sphere with suitably chosen simple factors without changing the topology.

Every CMC immersion of a Riemann surface M induces a 'monodromy representation' $\chi = \chi(\gamma,\, \lambda)$ of the fundamental group $\pi_1(M)$ of M with values in a unitary loop group. Moreover, as a function of the loop parameter λ, the monodromy matrices are holomorphic on $\mathbb{C}^*$. Dressing a given immersion with some arbitrary matrix will in general destroy the topology and the dressed surface will have a trivial fundamental group. Thus the question is: For which dressing matrices will the dressed immersion have the same fundamental group?

To explain this in more detail we note that in our method we associate with a given CMC immersion $f : M \to \mathbb{R}^3$ its associated family $f_\lambda : \widetilde{M} \to \mathbb{R}^3$, where $\widetilde{M}$ denotes the universal cover of M and $\lambda \in S^1$. This way we obtain the extended frame $F = F(z,\, \bar{z},\, \lambda)$ for $\lambda \in S^1$ and $z \in \widetilde{M}$. If $\gamma \in \pi_1(M)$ and we also denote the corresponding deck transformation by γ and write $\gamma^* F = F(\gamma(z),\, \overline{\gamma(z)},\, \lambda)$ then we have $\gamma^* F = \chi\, F\, k$ for some smooth $k : \widetilde{M} \to \mathbf{U}(1)$. Since, by assumption, the immersion $f_1 = f$ descends to M, the monodromy matrices satisfy the two closing conditions $\chi(\gamma,\, 1) = \pm \mathrm{Id}$ and $\partial_\lambda \chi|_{\lambda=1} = 0$ for all $\gamma \in \pi_1(M)$.

If $h = h(\lambda)$ is some dressing element and $\hat{F}$ denotes the dressed extended frame and we write $hF = \hat{F} V_+$ for the dressing equation, then $\gamma^* \hat{F} = h\, \chi\, h^{-1}\, \hat{F}\, V_+\, \gamma^* V_+^{-1}$. Thus $\hat{F}$ has some monodromy $\hat{\chi}$ if and only if $\hat{L}\hat{F} = \hat{F}\hat{W}_+$, where $\hat{W}_+ = V_+\, \gamma^* V_+^{-1}$ and $\hat{L} = (h\chi h^{-1})^{-1}\hat{\chi}$. As a consequence, $\hat{\chi} = h\chi h^{-1}\hat{L} = h\chi L h^{-1}$, where $L = h^{-1}\hat{L}h$.

We prove that a dressing matrix h preserves the fundamental group of an immersion if and only if for every $\gamma \in \pi_1(M)$ there exists some matrix $L = L(\gamma,\, \lambda)$, such that

(i) $LF = FW_+$,

(ii) $h\chi h^{-1}$ is holomorphic for $\lambda \in \mathbb{C}^*$,

(iii) $h\chi L h^{-1}$ is unitary on S^1, and

(iv) The closing conditions are satisfied for $h\chi L h^{-1}$.

Clearly, the situation simplifies considerably, if the original immersion has an umbilic

point, since then $L = \pm\mathrm{Id}$ [7]. Let us assume this for now. Considering in this case condition (ii) we see that χ is holomorphic for $\lambda \in \mathbb{C}^*$ and also $h\chi h^{-1}$ needs to be holomorphic on $\mathbb{C}^*$, in spite of the fact that h may only be defined on some circle of radius $0 < r \leq 1$. We prove that in the situation discussed here we can assume that $h = \mathcal{M}\,\mathcal{C}$, where $[\,\mathcal{C},\, \chi\,] = 0$ and $\mathcal{M}$ is meromorphic on some open dense subset $\mathbb{S} \subset \mathbb{C}^*$, which contains an open annulus about S^1. Moreover, $\mathbb{S}$ is solely defined by the eigenvalues of χ. In view of condition (iii) we further show that actually $\mathcal{M}$ can be chosen so that it is unitary on S^1.

This gives altogether a fairly complete description for the case, where an umbilic point exists. Similarly complete is the case, dealt with in Theorem 4.3, where χ is the monodromy of the standard round cylinder. For the general umbilic free case we have, at this point, only some partial results.

Let us briefly outline the contents of this paper. The first chapter sets the scene by defining the relevant loop groups, dressing and gauge actions and recalls the DPW representation [10] of CMC surfaces. In the second chapter we consider how automorphisms affect maps at the various levels of the DPW construction as a prerequisite to understanding monodromy. In chapter 3 we present a factorization as a necessary condition on the dressing matrix to ensure that the dressed surface retains the topology of the original surface. In the fourth chapter we apply our methods to the dressing orbit of the vacuum and give a self contained account of twisted simple factors. We conclude this work by discussing how dressing with simple factors is related to the monodromy and apply these methods to a class of CMC cylinders with umbilics.

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1 Notation and basic results

We begin by collecting some well known results on loop groups and the dressing action. We shall use the following notation for diagonal and off–diagonal matrices:

$$\mathrm{diag}[u,\, v] = \left(\begin{smallmatrix} u & 0 \\ 0 & v \end{smallmatrix}\right),\ \mathrm{off}[u,\, v] = \left(\begin{smallmatrix} 0 & u \\ v & 0 \end{smallmatrix}\right).$$

1.1 Loops. For real $r \in (0, 1]$, let $C_r = \{\lambda \in \mathbb{C} : |\lambda| = r\}$ and denote the r–Loop group of $\mathbf{SL}(2, \mathbb{C})$ by

$$\Lambda_r\mathbf{SL}(2, \mathbb{C}) = \{g : C_r \to \mathbf{SL}(2, \mathbb{C}) \text{ smooth}\}\,.$$

We have an involution on maps $C_r \to \mathbf{gl}(2, \mathbb{C})$ defined by

$$\sigma : g(\lambda) \mapsto \sigma_3 g(-\lambda)\sigma_3^{-1},\ \sigma_3 = \mathrm{diag}[1, -1], \tag{1.1.1}$$

and denote the *twisted* r–Loop group of $\mathbf{SL}(2, \mathbb{C})$ by

$$\Lambda_r\mathbf{SL}(2, \mathbb{C})_\sigma = \{g \in \Lambda_r\mathbf{SL}(2, \mathbb{C}) :\ \sigma g = g\}\,.$$

Analogously, one can define the Lie algebras of these groups, denoted by $\Lambda_r\mathbf{sl}(2,\mathbb{C})_\sigma$. To make these loop groups complex Banach Lie groups, we equip them, as in [10], with some H^s topology for $s > 1/2$ or some (possibly weighted) Wiener topology. Elements of these twisted loop groups are matrices whose off–diagonal entries are odd functions, while the diagonal entries are even functions of the parameter λ. All entries are in the Banach algebra $\mathcal{A}_r$ of H^s–smooth functions or of finite (possibly weighted) Wiener norm on C_r. Furthermore, we will use the following subgroups of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$: Let $I_r = \{\lambda \in \mathbb{C} : |\lambda| < r\}$ and denote

$$\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma = \{g \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma : g \text{ extends analytically to } I_r\}\,.$$

Let $A_r = \{\lambda \in \mathbb{C} : r < |\lambda| < 1/r\}$, and by abuse of notation we denote

$$\Lambda_r\mathbf{SU}(2)_\sigma = \{g \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma : g \text{ extends analytically to } A_r \text{ and } g|_{S^1} \in \mathbf{SU}(2)\}\,.$$

Let $g : A_r \to \mathbf{SL}(2,\mathbb{C})_\sigma$ be analytic. Then $g \in \Lambda_r\mathbf{SU}(2)_\sigma$ if and only if $\varrho\, g = g$ for

$$(\varrho g)(\lambda) := \overline{g(1/\bar{\lambda})}^{t-1}. \tag{1.1.2}$$

or alternatively, $g \in \Lambda_r\mathbf{SU}(2)_\sigma$ if and only if

$$(g^*)(\lambda) := \overline{g(1/\bar{\lambda})}^{t} = g(\lambda)^{-1}. \tag{1.1.3}$$

For $r = 1$ we will always omit the subscript 'r'. All the groups above are Banach Lie subgroups of $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$. The corresponding Banach Lie sub algebras of $\Lambda_r\mathbf{sl}(2,\mathbb{C})_\sigma$ are defined analogously.

1.2 Iwasawa decomposition. Of fundamental importance in our investigation is a certain Loop group factorization. We modify a result from [20] to obtain:

Multiplication $\Lambda_r\mathbf{SU}(2)_\sigma \times \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ is a real analytic surjection. An associated splitting $g = FB$ of $g \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ with $F \in \Lambda_r\mathbf{SU}(2)_\sigma$ and $B \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ will be called an Iwasawa decomposition of g. Note that

$$\Lambda_r\mathbf{SU}(2)_\sigma \cap \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma = \mathbf{U}(1). \tag{1.2.1}$$

If we demand that $B(\lambda = 0)$ has positive real diagonal entries, then the multiplication map above is a real analytic diffeomorphism onto. We will call this decomposition the "unique" Iwasawa decomposition.

1.3 Untwisted loops. We shall be primarily working with the twisted loop groups and algebras. On occasion it is advantageous to switch to the untwisted setting via the isomorphism between untwisted and twisted r–loops: let $X_u \in \Lambda_r\mathbf{SL}(2,\mathbb{C})$ be an untwisted loop. Then the corresponding twisted loop is given by $X_t(\lambda) = D\,X_u(\lambda^2)\,D^{-1}$ where $D = \mathrm{diag}[\sqrt{\lambda},\, 1/\sqrt{\lambda}]$. In matrix notation, this isomorphism is given by

$$\begin{pmatrix} a(\lambda) & b(\lambda) \\ c(\lambda) & d(\lambda) \end{pmatrix} \cong \begin{pmatrix} a(\lambda^2) & \lambda\, b(\lambda^2) \\ \lambda^{-1}c(\lambda^2) & d(\lambda^2) \end{pmatrix}. \tag{1.3.1}$$

One technical issue in switching between untwisted and twisted loops is that under this twisting isomorphism, an untwisted positive loop is not automatically positive when twisted, since the untwisted loop at $\lambda = 0$ need not be upper triangular. To circumvent this issue, we must first perform a Gram-Schmidt orthonormalization at $\lambda = 0$ before twisting the loop.

More precisely, consider an untwisted loop $\Phi_u \in \Lambda_r\mathbf{SL}(2,\mathbb{C})$ with an r-Iwasawa decomposition $\Phi_u = F_u\, B_u$ with untwisted $F_u \in \Lambda_r\mathbf{SU}(2)$ and $B_u \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})$. Now consider the twisted loop $\Phi_t(\lambda) = D\,\Phi_u(\lambda^2)\,D^{-1}$ obtained from Φ_u via the isomorphism (1.3.1) and let $\Phi_t = F_t\, B_t$ be an r-Iwasawa decomposition with twisted $F_t \in \Lambda_r\mathbf{SU}(2)_\sigma$ and $B_t \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. If B_u has an expansion $B_u = B_0 + \lambda B_1 + \ldots$, then as $B_0 \in \mathbf{SL}(2,\mathbb{C})$, we may write $B_0 = Q\,R$ via Gram-Schmidt orthonormalization with $Q \in \mathbf{SU}(2)$ and $R \in \mathbf{SL}(2,\mathbb{C})$ upper triangular. Consequently, $Q^{-1}B_u \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})$ and $Q^{-1}B_u|_{\lambda=0} = R$ and thus $D\,Q^{-1}B_u(\lambda^2)\,D^{-1} \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Furthermore, we then have that

$$\Phi_t(\lambda) = \left(D\,F_u(\lambda^2)\,Q\,D^{-1}\right)\left(D\,Q^{-1}B_u(\lambda^2)\,D^{-1}\right) \tag{1.3.2}$$

with $D\,F_u(\lambda^2)\,Q\,D^{-1} \in \Lambda_r\mathbf{SU}(2)_\sigma$ and $D\,Q^{-1}B_u(\lambda^2)\,D^{-1} \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ is an r-Iwasawa decomposition. It is in this sense that we may carry an Iwasawa decomposition in the untwisted setting over to the twisted setting.

1.4 DPW method. Let $\Omega(M)$ denote the holomorphic 1–forms on a Riemann surface M and define the

$$\Lambda\Omega(M) = \Omega(M) \otimes \left\{ \xi : \mathbb{C}^* \to \mathbf{sl}(2,\mathbb{C}) \text{ holomorphic} : \xi(\lambda) = \sum_{j \geq -1} \xi_j \lambda^j,\ \sigma\xi = \xi \right\}.$$

CMC surfaces come in S^1 families, the *associated family*. The DPW representation [10] constructs all conformal CMC immersions of the universal cover $\widetilde{M}$ in the following three steps: Let $\xi \in \Lambda\Omega(\widetilde{M})$, $\tilde{z}_0 \in \widetilde{M}$ and $\Phi_0 \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ for some $r \in (0,1]$. To avoid totally umbilical surfaces, we assume $\det \xi_{-1} \not\equiv 0$.

1. Solve the initial value problem

$$d\Phi = \Phi\xi,\ \Phi(\tilde{z}_0) = \Phi_0 \tag{1.4.1}$$

to obtain a unique *holomorphic frame* $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$.

2. Iwasawa decompose the map $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ point-wise on $\widetilde{M}$

$$\Phi(z,\,\lambda) = F(z,\,\bar{z},\,\lambda)\,B(z,\,\bar{z},\,\lambda) \tag{1.4.2}$$

We will always assume that the factors F and B in (1.4.2) are real analytic in z. The map $F : \widetilde{M} \to \Lambda_r\mathbf{SU}(2)_\sigma$ will be called *unitary frame*. If we use the unique Iwasawa decomposition, then the factors F and B are automatically real analytic.

3. Let $H \in \mathbb{R}^*$ and $\partial_\lambda = \frac{\partial}{\partial\lambda}$. Plug F into the Sym-Bobenko formula

$$f_\lambda = -\tfrac{1}{2H}\left(i\lambda \tfrac{\partial F}{\partial \lambda}\,F^{-1} + \tfrac{i}{2}F\sigma_3F^{-1}\right) \tag{1.4.3}$$

to obtain (possibly branched) conformal CMC immersions $f_\lambda : \widetilde{M} \to \Lambda\mathbf{su}(2)_\sigma$ with mean curvature H, that is, for each $\lambda_0 \in S^1$ we have a conformal CMC immersion $f_{\lambda_0} : \widetilde{M} \to \mathbf{su}(2) \cong \mathbb{R}^3$. Note, $f_\lambda(z)$ is branched at z_0 if and only if for $\xi_{-1} = \mathrm{off}[a,\, b]$ we have $a(z_0) = 0$.

1.5 Frames. We call $\xi \in \Lambda\Omega(\widetilde{M})$ a *holomorphic potential*. If $\Phi_0 = \phi_u\phi_+$ is an Iwasawa decomposition, then it is not hard to see that the immersions obtained from $(\xi, \Phi_0, \tilde{z}_0)$ and $(\xi, \phi_+, \tilde{z}_0)$ differ by a λ-dependent rigid motion. Thus the choice of initial condition may be restricted to $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Further, if $\Phi_0 \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ then $F(\tilde{z}_0) \in \mathbf{U}(1)$ and $\Phi = FF(\tilde{z}_0)^{-1}F(\tilde{z}_0)B$. Hence we may assume without loss of generality that

$$F(\tilde{z}_0) \equiv \mathrm{Id} \text{ for all } \lambda \in S^1. \tag{1.5.1}$$

Let $\mathbf{sl}(2,\mathbb{C}) = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition induced by the involution σ defined in (1.1.1). The specific form of ξ ensures that the Maurer–Cartan form of F acquires the form

$$F^{-1}dF = \lambda^{-1}\alpha_\mathfrak{p}' + \alpha_\mathfrak{k} + \lambda\,\alpha_\mathfrak{p}'' \tag{1.5.2}$$

where $\alpha_\mathfrak{p}'$ and $\alpha_\mathfrak{p}''$ are (1, 0) respectively $(0,1)$-forms on $\widetilde{M}$ with values in $\mathfrak{p}$ and $\alpha_\mathfrak{k}$ takes values in $\mathfrak{k}$. For each $p \in \widetilde{M}$, $\lambda \mapsto F(p,\lambda)$ is holomorphic on $\mathbb{C}^*$. Maps

$$F : \widetilde{M} \to \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma := \bigcap_{r\in(0,1]} \Lambda_r\mathbf{SU}(2)_\sigma \tag{1.5.3}$$

with the property (1.5.2) are called *extended unitary frames* and denoted by $\mathcal{F}(\widetilde{M})$. We call the extended unitary frames for which (1.5.1) holds the *normalized extended unitary frames* and denote these by

$$\mathcal{F}_{\mathrm{Id}}(\widetilde{M}) = \left\{F \in \mathcal{F}(\widetilde{M}) : F(\tilde{z}_0) = \mathrm{Id} \text{ for some } \tilde{z}_0 \in \widetilde{M}\right\}. \tag{1.5.4}$$

1.6 Gauge. The DPW representation $(\xi, \Phi_0, \tilde{z}_0) \mapsto \mathcal{F}(\widetilde{M})$ is a surjective map [10]. Injectivity fails since the *Gauge group*

$$\mathcal{G}_r(\widetilde{M}) = \{G : \widetilde{M} \to \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma \text{ holomorphic }\}, \tag{1.6.1}$$

acts by right multiplication on the fibers of this map. On the level of the potential, this *gauge action* is given by

$$\xi.G = G^{-1}\xi\; G + G^{-1}dG. \tag{1.6.2}$$

A computation shows that if Φ solves (1.4.1) with triple $(\xi, \Phi_0, \tilde{z}_0)$ and $G \in \mathcal{G}_r(\widetilde{M})$ then the triples $(\xi, \Phi_0, \tilde{z}_0)$ and $(\xi.G, \Phi_0G(\tilde{z}_0), \tilde{z}_0)$ induce the same CMC immersions, assuming $H \neq 0$.

1.7 Dressing. For $r \in (0,1]$, $h \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ and $F_0 \in \mathcal{F}(\widetilde{M})$ let

$$hF_0 = F\,B \tag{1.7.1}$$

be a point-wise Iwasawa decomposition of hF_0 in $\Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, where every factor is analytic in z. Then $F \in \mathcal{F}(\widetilde{M})$.

Note that $F : \widetilde{M} \to \Lambda_r\mathbf{SU}(2)_\sigma$ and $B : \widetilde{M} \to \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ are not uniquely determined, since for any smooth $U : \widetilde{M} \to \mathbf{U}(1)$, we again have an Iwasawa decomposition given by $hF_0 = FU\,U^{-1}B$. We shall write

$$F \in [h\#F_0] \tag{1.7.2}$$

to signify that $F : \widetilde{M} \to \Lambda_r\mathbf{SU}(2)_\sigma$ satisfies (1.7.1). We call $[h\#F_0]$ the *dressing class* of F_0 under h and say that $F \in [h\#F_0]$ was obtained by *dressing* $F_0 \in \mathcal{F}(\widetilde{M})$ by $h \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$.

To preserve the base point condition (1.5.1), one needs to restrict to dressing with elements $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

Lemma: Let $h \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ and $F_0 \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ be a normalized extended unitary frame. Then there exists a normalized extended unitary frame in $[h\#F_0]$ if and only if $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

Proof: A straightforward computation, see Proposition 2.9 of [5], shows that the Maurer–Cartan form of any element in $[h\#F_0]$ is again of the form (1.5.2). The issue is that for $F_0 \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ we have $F_0(\tilde{z}_0) = \mathrm{Id}$, while for $F \in [h\#F_0]$ we have apriori only $F(\tilde{z}_0) \in \mathbf{U}(1)$.

If $hF_0 = FB$ is an Iwasawa decomposition, then $F_0(\tilde{z}_0) = F(\tilde{z}_0) = \mathrm{Id}$, implies $h = B(\tilde{z}_0) \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$.

Conversely, if $hF_0 = FB$ is an Iwasawa decomposition, then so is

$$hF_0 = FF(\tilde{z}_0)^{-1}F(\tilde{z}_0)B. \tag{1.7.3}$$

Hence $FF(\tilde{z}_0)^{-1} \in [h\#F_0]$ and satisfies (1.5.1). Thus $FF(\tilde{z}_0)^{-1} \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$. $\square$

We thus have a left action of $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ on $\mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ and shall write $F = h\#F_0$ to signify that $F \in [h\#F_0]$ with $F(\tilde{z}_0) = \mathrm{Id}$. Evaluating $hF_0 = h\#F_0\,B$ at $\tilde{z}_0$ also gives $h = B(\tilde{z}_0)$.

1.8 Isotropies. We denote the isotropy group of a map $F : \widetilde{M} \to \Lambda_r\mathbf{SU}(2)_\sigma$ under dressing by

$$\mathrm{Iso}_r(F) = \{h \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma : F \in [h\#F]\}\,. \tag{1.8.1}$$

For a holomorphic frame $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ we define the isotropy under dressing by

$$\mathrm{Iso}_r(\Phi) = \{g \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma : g\,\Phi = \Phi\,G \text{ for some } G \in \mathcal{G}_r(\widetilde{M})\}. \tag{1.8.2}$$

If $g \in \mathrm{Iso}_r(\Phi)$ with $g\Phi = \Phi G$ and $\Phi = F\,B$ is an Iwasawa decomposition of Φ, then $gF = FBGB^{-1}$ with $BGB^{-1} \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Hence $g \in \mathrm{Iso}_r(F)$. Conversely, if $h \in \mathrm{Iso}_r(F)$, and $hF = FH$ is an Iwasawa decomposition, then $h\Phi = \Phi B^{-1}HB$ and thus

$$\mathrm{Iso}_r(\Phi) = \mathrm{Iso}_r(F). \tag{1.8.3}$$

Further, using (1.2.1) it is straightforward to verify that

$$\mathrm{Iso}_r(F) \cap \Lambda_r \mathbf{SU}(2)_\sigma = \mathbf{U}(1) \text{ for } F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M}). \tag{1.8.4}$$

Umbilic points occur naturally on compact CMC surfaces of genus $g \geq 2$ as well as on complete, open CMC surfaces with more than two ends. In this context, we quote

Theorem [7]: If the associated family of $F \in \mathcal{F}(\widetilde{M})$ has umbilics, then $\mathrm{Iso}_r(F) = \{\pm \mathrm{Id}\}$ for all $r \in (0, 1]$. □

2 Symmetries

The notion of symmetry for CMC immersions has been discussed in the articles [6, 8]. It turns out that symmetry can be defined on various levels in the context of the DPW representation. The problem of dealing with symmetries in the DPW representation stem mostly from the fact that symmetries on the potential level are, by and large, defined as coming from symmetries on the immersion level and have not yet been defined completely intrinsically on the potential level. Since symmetries associated with automorphisms $Aut(\widetilde{M})$ of the universal cover are central to this article, we recall the basic facts, retaining the notation of section 1.4.

2.1 Definitions. Let $f : \widetilde{M} \to \Lambda \mathbf{su}(2)_\sigma$ be a CMC $H \neq 0$ immersion generated by a triple $(\xi, \Phi_0, \tilde{z}_0)$ with Φ its holomorphic frame and F its unitary frame. In the rest of this section we will always assume $\gamma \in Aut(\widetilde{M})$ and for a map G with domain $\widetilde{M}$ we shall write $\gamma^* G = G \circ \gamma$.

Immersion level: A triple $(\gamma, \mathbb{X}, T)$ with $\mathbb{X} \in \Lambda_r \mathbf{SU}(2)_\sigma$ and $T \in \Lambda_r \mathbf{su}(2)_\sigma$ is called a symmetry of f if and only if

$$\gamma^* f = \mathbb{X} \, f \, \mathbb{X}^{-1} + T. \tag{2.1.1}$$

Unitary frame level: A pair $(\gamma, \mathbb{X})$ with $\mathbb{X} \in \Lambda_r \mathbf{SU}(2)_\sigma$ is called a symmetry of F if and only if there exists a smooth map $k \in C^\infty(\widetilde{M}, \mathbf{U}(1))$ such that

$$\gamma^* F = \mathbb{X} \, F \, k. \tag{2.1.2}$$

Holomorphic frame level: A pair $(\gamma, \mathbb{X})$ with $\mathbb{X} \in \Lambda_r \mathbf{SU}(2)_\sigma$ is a symmetry of Φ if and only if there is a map $H \in \mathcal{G}_r(\widetilde{M})$ such that

$$\gamma^* \Phi = \mathbb{X} \, \Phi \, H. \tag{2.1.3}$$

Potential level: A pair (γ, G) with $G \in \mathcal{G}_r(\widetilde{M})$ is called a symmetry of ξ if and only if

$$\gamma^* \xi = \xi . G \, . \tag{2.1.4}$$

where $\xi.G$ denotes the gauge transformation of ξ by G as defined in (1.6.2).

Theorem: Let $f : \widetilde{M} \to \Lambda\mathbf{su}(2)_\sigma$ be a CMC $H \neq 0$ immersion generated by the triple $(\xi, \Phi_0, \tilde{z}_0)$ with Φ and F the holomorphic respectively unitary frame.

(i) If $(\gamma,\, \mathbb{X}_1,\, T_1)$ and $(\gamma,\, \mathbb{X}_2,\, T_2)$ are symmetries of f, then $\mathbb{X}_2 = \pm\mathbb{X}_1$ and $T_1 = T_2$.

(ii) If $(\gamma,\, \mathbb{X}_1)$ and $(\gamma,\, \mathbb{X}_2)$ are symmetries of F then $\mathbb{X}_2 = \pm\mathbb{X}_1$.

(iii) If $(\gamma,\, \mathbb{X}_1)$ and $(\gamma,\, \mathbb{X}_2)$ are symmetries of Φ then $\mathbb{X}_2 = \pm\mathbb{X}_1$.

Proof: (i) If $(\gamma,\, \mathbb{X}_1,\, T_1)$ and $(\gamma,\, \mathbb{X}_2,\, T_2)$ are symmetries, then $\gamma^* f = \mathbb{X}_1 f \mathbb{X}_1^{-1} + T_1 = \mathbb{X}_2 f \mathbb{X}_2^{-1} + T_2$ implies $\mathbb{X}\, f\, \mathbb{X}^{-1} + T = f$ with $\mathbb{X} = \mathbb{X}_2^{-1}\mathbb{X}_1$ and $T = \mathbb{X}_2^{-1}(T_1 - T_2)\mathbb{X}_2$. Since a CMC $H \neq 0$ surface in $\mathbb{R}^3$ is never contained in an affine plane, $\mathbb{X}\, f\, \mathbb{X}^{-1} + T = f$ implies $\mathbb{X} = \pm\mathrm{Id}$ and $T = 0$.

(ii) If $(\gamma,\, \mathbb{X}_1)$ and $(\gamma,\, \mathbb{X}_2)$ are symmetries of F, then $\gamma^* F = \mathbb{X}_1 F k_1 = \mathbb{X}_2 F k_2$. Plugging $\mathbb{X}_1 F k_1$ and $\mathbb{X}_2 F k_2$ into the Sym–Bobenko formula and equating yields

$$\mathbb{X}_1 f \mathbb{X}_1^{-1} - \tfrac{i\lambda}{2H}(\partial_\lambda \mathbb{X}_1)\mathbb{X}_1^{-1} = \mathbb{X}_2 f \mathbb{X}_2^{-1} - \tfrac{i\lambda}{2H}(\partial_\lambda \mathbb{X}_2)\mathbb{X}_2^{-1}. \tag{2.1.5}$$

Now $\mathbb{X}_2^{-1}\mathbb{X}_1\, f\, \mathbb{X}_1^{-1}\mathbb{X}_2 + \frac{i\lambda}{2H}\mathbb{X}_2^{-1}\left((\partial_\lambda \mathbb{X}_2)\mathbb{X}_2^{-1} - (\partial_\lambda \mathbb{X}_1)\mathbb{X}_1^{-1}\right)\mathbb{X}_2 = f$ and part (i) implies $\mathbb{X}_2 = \pm\mathbb{X}_1$.

(iii) If $(\gamma,\, \mathbb{X}_1)$ and $(\gamma,\, \mathbb{X}_2)$ are symmetries of Φ then $\gamma^*\Phi = \mathbb{X}_1 \Phi H = \mathbb{X}_2 \Phi G$ implies $\mathbb{X}_2^{-1}\mathbb{X}_1 \Phi = \Phi G H^{-1}$ and shows that $\mathbb{X}_2^{-1}\mathbb{X}_1 \in \mathrm{Iso}(\Phi)$. Let $\Phi = FB$ be an Iwasawa decomposition of Φ. Then $\mathbb{X}_2^{-1}\mathbb{X}_1 \in \mathrm{Iso}(F)$, since $\mathrm{Iso}(\Phi) = \mathrm{Iso}(F)$ by (1.8.3). Hence $\mathbb{X}_2^{-1}\mathbb{X}_1 = \pm\mathrm{Id}$ by (1.8.4). □

For symmetries $(\gamma,\, \mathbb{X}_\gamma)$ and $(\mu,\, \mathbb{X}_\mu)$ of $F \in \mathcal{F}(\widetilde{M})$ and corresponding maps $k_\gamma,\, k_\mu$, we have $\mu^*\, \gamma^*\, F = \mathbb{X}_\gamma\, \mathbb{X}_\mu\, F\, k_\mu\, \mu^* k_\gamma$ and if we assume that $((\gamma \circ \mu),\, \mathbb{X}_{\gamma\mu})$ is a symmetry, then part (ii) of Theorem 2.1 implies $\mathbb{X}_{\gamma\mu} = \pm\mathbb{X}_\gamma\, \mathbb{X}_\mu$ and consequently

$$k_{\gamma\mu} = \pm k_\mu\, \mu^* k_\gamma. \tag{2.1.6}$$

2.2 Symmetries of Φ. We briefly investigate how symmetries of a holomorphic potential descend to symmetries of the corresponding holomorphic frame and note a simple consequence in case the potential is invariant. In view of these results and our subsequent inquiry into the relationship with the dressing action, we allow symmetries on the level of holomorphic frames of the form (γ, X) with $X \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$.

Proposition: Let $\gamma \in Aut(\widetilde{M})$ and $(\gamma,\, G)$ be a symmetry of $\xi \in \Lambda\Omega(\widetilde{M})$. Let $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ be a solution to $d\Phi = \Phi\xi$. Then there exists $X \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ such that $\gamma^*\Phi = X\Phi G$.

Proof: Since $\gamma^*\Phi$ and ΦG both solve the differential equation $dY = Y\gamma^*\xi$, there exists $X \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ with $\gamma^*\Phi = X\Phi G$. More explicitly, if Φ solves $d\Phi = \Phi\xi$, $\Phi(\tilde{z}_0) = \Phi_0$, then $X = \Phi(\gamma(\tilde{z}_0))G(\tilde{z}_0)^{-1}\Phi_0^{-1}$. □

We shall again call a pair $(\gamma,\, X)$ with $\gamma \in Aut(\widetilde{M})$ and $X \in \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ a symmetry of a map $\Phi : \widetilde{M} \to \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ if there exists a $G \in \mathcal{G}_r(\widetilde{M})$ such that $\gamma^*\Phi = X\,\Phi\, G$.

Corollary: Let $\gamma \in Aut(\widetilde{M})$ and $\xi \in \Lambda\Omega(\widetilde{M})$ with $\gamma^*\xi = \xi$. Let Φ be a solution to $d\Phi = \Phi\xi$. Then there exists $X \in \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ such that $\gamma^*\Phi = X\Phi$. □

2.3 Symmetries of F. In view of the previous section, we now characterize how symmetries of the holomorphic frame descend to symmetries of the corresponding extended unitary frame and show that for open Riemann surfaces, the co-cycle factor in equation (2.1.2) can be gauged away.

Lemma: Let $(\gamma,\, X)$ be a symmetry of $\Phi : \widetilde{M} \to \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$. Let $\Phi = F\,B$ be an Iwasawa decomposition of Φ. Then there exists an $\mathbb{X} \in \Lambda_r \mathbf{SU}(2)_\sigma$ such that $(\gamma, \mathbb{X})$ is a symmetry of F if and only if there exists an element $L \in \mathrm{Iso}_r(\Phi)$ with $LX^{-1} \in \Lambda_r \mathbf{SU}(2)_\sigma$.

Proof: Let $G \in \mathcal{G}_r(\widetilde{M})$ such that $\gamma^*\Phi = X\Phi G$. If $(\gamma, \mathbb{X})$ with $\mathbb{X} \in \Lambda_r \mathbf{SU}(2)_\sigma$ is a symmetry of F, then by (2.1.2) there exists a differentiable map $k : \widetilde{M} \to \mathbf{U}(1)$ such that $\gamma^*F = \mathbb{X}Fk$. In combination with $F = \Phi B^{-1}$, after rearranging, we obtain

$$\mathbb{X}^{-1} X\,\Phi = \Phi B^{-1} k \gamma^* B\, G^{-1}. \tag{2.3.1}$$

Then $L := \mathbb{X}^{-1}X \in \mathrm{Iso}_r(\Phi)$ as $B^{-1}k\gamma^*B\,G^{-1} \in \mathcal{G}_r(\widetilde{M})$ and $L\,X^{-1} = \mathbb{X}^{-1} \in \Lambda_r \mathbf{SU}(2)_\sigma$. Conversely, let $L \in \mathrm{Iso}_r(\Phi)$ and $L\,X^{-1} \in \Lambda_r \mathbf{SU}(2)_\sigma$. Then there exists an element $H \in \mathcal{G}_r(\widetilde{M})$ with $L\,\Phi = \Phi H$. By (1.8.3), $L \in \mathrm{Iso}_r(F)$ and $LF = FV$ for $V = BHB^{-1} \in \mathcal{G}_r(\widetilde{M})$. Then $\gamma^*\Phi = X\Phi G$ yields

$$\gamma^* F = X\,F\,B\,G\,\gamma^* B^{-1} = X\,L^{-1} F\,V\,B\,G\,\gamma^* B^{-1}. \tag{2.3.2}$$

Define $\mathbb{X} := X\,L^{-1}$ and $k := VBG\gamma^*B^{-1}$. On the one hand, $k \in \mathcal{G}_r(\widetilde{M})$ while on the other hand $k = F^{-1}\mathbb{X}^{-1}\gamma^*F$ takes values in $\Lambda_r \mathbf{SU}(2)_\sigma$, since $\mathbb{X} \in \Lambda_r \mathbf{SU}(2)_\sigma$ by assumption. From equation (1.2.1) we conclude that $k : \widetilde{M} \to \mathbf{U}(1)$. □

Theorem: Let M be an open Riemann surface with Fuchsian group Γ and let $f_\lambda : \widetilde{M} \to \mathbb{R}^3$ be an associated family of CMC $H \neq 0$ immersions. Then there exists an extended unitary frame $F \in \mathcal{F}(\widetilde{M})$ for f_λ such that for every $\gamma \in \Gamma$ there exists $\mathbb{X} \in \Lambda_{\mathbb{C}^*} \mathbf{SU}(2)_\sigma$ such that $\gamma^*F = \mathbb{X}\,F$.

Proof: First we apply [8], Theorem 2.3 and infer that there exists some extended frame $\tilde{F} \in \mathcal{F}(\widetilde{M})$ for f_λ such that for every $\gamma \in \Gamma$ we have $\gamma^*\tilde{F} = \mathbb{X}\,\tilde{F}\,k$, with $\mathbb{X} \in \Lambda_{\mathbb{C}^*} \mathbf{SU}(2)_\sigma$ and $k : \widetilde{M} \to \mathbf{U}(1)$ as in (2.1.2). Writing $\tilde{F} = \Phi\,B$ with $\Phi : \widetilde{M} \to \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ holomorphic and $B \in \mathcal{G}_r(\widetilde{M})$, we obtain $\gamma^*\Phi = \mathbb{X}\,\Phi\,W$ with $W \in \mathcal{G}_r(\widetilde{M})$. It is easy to see that W satisfies the co-cycle condition. Therefore, since the open Riemann surface $\widetilde{M}$ is Stein, by [3], W is a co-boundary and whence $W = W\,\gamma^* W^{-1}$. Hence $\tilde{\Phi} = \Phi\,W$ satisfies $\gamma^*\tilde{\Phi} = \mathbb{X}\,\tilde{\Phi}$. Iwasawa splitting $\tilde{\Phi} = F\hat{B}$ with $\hat{B}$ normalized such that the λ^0 coefficient in $\hat{B}$ has positive real entries we obtain $\gamma^*F = \mathbb{X}\,F$. □

2.4 Symmetries & Dressing. Next, we characterize how symmetries between two dressing equivalent unitary frames are related.

Theorem: Let $F_0 \in \mathcal{F}(\widetilde{M})$ have symmetry $(\gamma, \mathbb{X}_0)$ and $h \in \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$. Then $F \in [h \# F_0]$ has symmetry $(\gamma, \mathbb{X})$ if and only if there exists a $L \in \mathrm{Iso}_r(F)$ such that $\mathbb{X} = h\mathbb{X}_0 h^{-1} L^{-1}$. Furthermore, if both $F_0,\, F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$, then $L \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$.

Proof: Writing $hF_0 = FB$ for an Iwasawa decomposition and using $\gamma^* F_0 = \mathbb{X}_0 F_0 k_0$, we obtain $h\mathbb{X}_0 h^{-1} F = \gamma^* F \gamma^* B k_0^{-1} B^{-1}$. If $(\gamma, \mathbb{X})$ is a symmetry of F then $\gamma^* F = \mathbb{X} F k$, whence

$$\mathbb{X}^{-1} h \mathbb{X}_0 h^{-1} F = F k \gamma^* B k_0^{-1} B^{-1}. \tag{2.4.1}$$

Set $L := \mathbb{X}^{-1} h \mathbb{X}_0 h^{-1}$ and $H := k\gamma^* B k_0^{-1} B^{-1}$. Clearly $L \in \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$ and $H \in \mathcal{G}_r(\widetilde{M})$, so equation (2.4.1) reads $LF = FH$ and is an Iwasawa decomposition of LF. Hence $L \in \mathrm{Iso}_r(F)$, and by construction $\mathbb{X} = h\mathbb{X}_0 h^{-1} L^{-1}$.

Conversely, assume there exists a map $L \in \mathrm{Iso}_r(F)$ such that $LF = FH$ for some $H \in \mathcal{G}_r(\widetilde{M})$ and $h\mathbb{X}_0 h^{-1} L^{-1} \in \Lambda_{\mathbb{C}^*} \mathbf{SU}(2)_\sigma$. Defining $\mathbb{X} = h\mathbb{X}_0 h^{-1} L^{-1}$ and $k = HBk_0 \gamma^* B^{-1}$, a computation yields $\gamma^* F = \mathbb{X} F k$. A priori, $k : \widetilde{M} \to \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$, but as $\gamma^* F$, $\mathbb{X}$ and F take values in $\Lambda_{\mathbb{C}^*} \mathbf{SU}(2)_\sigma$ so does k. Hence $k : \widetilde{M} \to \mathbf{U}(1)$ and consequently $(\gamma, \mathbb{X})$ is a symmetry of F. $\square$

Corollary: If in addition to the assumptions of Theorem 2.4, $\mathrm{Iso}_r(F) = \{\pm \mathrm{Id}\}$, then $(\gamma, \mathbb{X})$ is a symmetry of F if and only if $\mathbb{X} = \pm h\, \mathbb{X}_0 h^{-1}$.

Proof: The assumption $\mathrm{Iso}_r(F) = \{\pm \mathrm{Id}\}$ implies that for the maps L, H in the proof of Theorem 2.4 we have $L = \mathrm{Id}$ and $H \equiv \mathrm{Id}$, yielding the claim. $\square$

2.5 Groups of symmetries. Extending the results of section 2.4 to groups $\Gamma \subset Aut(\widetilde{M})$ of symmetries one obtains:

Theorem: Let $F_0 \in \mathcal{F}_r(\widetilde{M})$ admit symmetries $(\gamma, \mathbb{X}_0(\gamma))$ for all $\gamma \in \Gamma$ and let $h \in \Lambda_r \mathbf{SL}(2,\mathbb{C})_\sigma$. Then $F \in [h \# F_0]$ has symmetries $(\gamma, \mathbb{X}(\gamma))$ for all $\gamma \in \Gamma$ if and only if there exist maps $L(\gamma) \in \mathrm{Iso}_r(F)$ such that $\mathbb{X}(\gamma) = h\mathbb{X}_0(\gamma) h^{-1} L(\gamma)^{-1}$. Moreover, if $\mathrm{Iso}_r(F) = \{\pm \mathrm{Id}\}$, then F admits symmetries $(\gamma, \mathbb{X}(\gamma))$ for all $\gamma \in \Gamma$ if and only if $\mathbb{X}(\gamma) = h\mathbb{X}_0(\gamma) h^{-1}$ for all $\gamma \in \Gamma$. $\square$

2.6 Symmetries & Isotropy Let $(\gamma, \mathbb{X})$ be a symmetry of $F \in \mathcal{F}(\widetilde{M})$ and $\gamma^* F = \mathbb{X} F k$. If $L \in \mathrm{Iso}_r(F)$ and $LF = FB$ for an Iwasawa decomposition, then $\mathbb{X}^{-1} L \mathbb{X} F = F k \gamma^* B k^{-1}$. As $(k\gamma^* B k^{-1})$ is $\Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$–valued, $\mathbb{X}^{-1} L \mathbb{X} \in \mathrm{Iso}_r(F)$ and every symmetry $(\gamma, \mathbb{X})$ induces an inner automorphism $S : \mathrm{Iso}_r(F) \to \mathrm{Iso}_r(F)$, $L \mapsto \mathbb{X}^{-1} L \mathbb{X}$.

Now consider $F_0 \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$, $h \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ and let $F = h \# F_0$. Assume that $(\gamma,\, \mathbb{X}_0)$ and $(\gamma,\, \mathbb{X})$ are symmetries of F_0 respectively F. If further $h \in \mathrm{Iso}_r(F)$, then we obtain a map $S : \mathrm{Iso}_r(F) \to \mathrm{Iso}_r(F)$, $h \mapsto \mathbb{X}^{-1} h \mathbb{X}_0 h^{-1}$ with the property $S(h_1 h_2) = S(h_1) h_1 S(h_2) h_1^{-1}$.

3 Invariant potentials & Monodromy

We start our investigation of the relationship between monodromy and dressing by defining the notion of monodromy on the level of holomorphic and unitary frames in the DPW framework.

Definition: Let M be a connected Riemann surface with universal cover $\widetilde{M}$ and Δ the group of deck transformations. Let $\xi \in \Lambda\Omega(\widetilde{M})$ and $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ be a solution of $d\Phi = \Phi\xi$ and $\tau \in \Delta$. A loop $\varrho(\tau) \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ such that $(\tau,\, \varrho(\tau))$ is a symmetry of Φ will be called a *monodromy* of Φ with respect to τ. Let $\Phi = F\,B$ be an Iwasawa decomposition. A loop $\chi(\tau) \in \Lambda_r\mathbf{SU}(2)_\sigma$ such that $(\tau,\, \chi(\tau))$ is a symmetry of F is called a *monodromy* of F with respect to τ.

Remark: A monodromy of a normalized extended unitary frame $F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ takes values in $\Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$.

3.1 Existence of monodromies. Apriori, it is not clear that for some $\tau \in \Delta$ there exist monodromies $\varrho(\tau)$ and $\chi(\tau)$ as in the definitions above. In case the corresponding potentials are invariant under Δ we can evoke Corollary 2.2 and obtain the following sufficient conditions for the existence of monodromies.

Proposition: (i) Let $\xi \in \Lambda\Omega(\widetilde{M})$ and Φ be a solution of $d\Phi = \Phi\,\xi$ with initial condition $\Phi_0 \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$. Let $\tau \in \Delta$ with $\tau^*\xi = \xi$. Then there exists a monodromy $\varrho(\tau) \in \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ such that $\tau^*\Phi = \varrho(\tau)\,\Phi$.

(ii) Let $F : \widetilde{M} \to \Lambda_r\mathbf{SU}(2)_\sigma$ and $\alpha = F^{-1}dF$. If $\tau^*\alpha = \alpha$ for $\tau \in \Delta$, then there exists a monodromy $\chi(\tau) \in \Lambda_r\mathbf{SU}(2)_\sigma$ such that $\tau^*F = \chi(\tau)\,F$. □

It is shown in [9] that CMC immersions of open Riemann surfaces M can always be generated by Δ–invariant potentials $\xi \in \Lambda\Omega(\widetilde{M})$. If $\Phi = FB$ is the Iwasawa decomposition of a holomorphic frame $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$, then we shall need to study the monodromy of F in order to control the periodicity of the resulting immersion (1.4.3). Even if F is obtained from a potential that is invariant under Δ, we are a priori not assured that there exist loops $\chi(\tau) \in \Lambda_r\mathbf{SU}(2)_\sigma$ and maps $k(\tau) : \widetilde{M} \to \mathbf{U}(1)$ such that equation (2.1.2) holds for all $\tau \in \Delta$ or even more strongly, that for $\alpha = F^{-1}dF$ we have $\tau^*\alpha = \alpha$ for all $\tau \in \Delta$. In analogy to Theorem 2.4, this issue is characterized by the following

Lemma: Let $(\xi, \Phi_0, \tilde{z}_0)$ generate the holomorphic frame $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ and extended unitary frame $F \in \mathcal{F}(\widetilde{M})$. Let $\tau \in \Delta$ and $\varrho(\tau)$ be a monodromy of Φ. Then the following are equivalent.

(i) There exists a monodromy $\chi(\tau) \in \Lambda_r\mathbf{SU}(2)_\sigma$ of F with respect to τ.

(ii) There exists an element $L(\tau) \in \mathrm{Iso}_r(\Phi)$ with $L(\tau)\varrho^{-1}(\tau) \in \Lambda_r\mathbf{SU}(2)_\sigma$.

Proof: We first show that (i) implies (ii). Let $\chi(\tau) \in \Lambda_r \mathbf{SU}(2)_\sigma$ and $k(\tau) : \widetilde{M} \to \mathbf{U}(1)$ such that $\tau^* F = \chi(\tau) F\, k(\tau)$. Using $F = \Phi B^{-1}$ and $\tau^*\Phi = \varrho(\tau)\Phi$ and rearranging, we obtain $\chi(\tau)^{-1}\varrho(\tau)\,\Phi = \Phi\, B^{-1} k(\tau)\,\tau^* B$. Then $L(\tau) := \chi(\tau)^{-1}\varrho(\tau) \in \mathrm{Iso}_r(\Phi)$ and $L(\tau)\varrho(\tau)^{-1} \in \Lambda_r\mathbf{SU}(2)_\sigma$.

Conversely, let $L(\tau) \in \mathrm{Iso}_r(\Phi)$ and assume $L(\tau)\varrho(\tau)^{-1} \in \Lambda_r\mathbf{SU}(2)_\sigma$. Then there exists an element $G(\tau) \in \mathcal{G}_r(\widetilde{M})$ with $L(\tau)\,\Phi = \Phi\, G(\tau)$. Multiplying this last equation on the right by $\tau^*\Phi^{-1}$ and rearranging gives

$$F^{-1} L(\tau)\, \varrho(\tau)^{-1}\, \tau^* F = B\, G(\tau)\, \tau^* B^{-1}. \tag{3.1.1}$$

The left respectively right hand side of (3.1.1) takes values in $\Lambda_r\mathbf{SU}(2)_\sigma$ respectively $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Hence, by (1.2.1), both sides are λ–independent and $\mathbf{U}(1)$–valued. Set $k(\tau) := F^{-1}L(\tau)\,\varrho(\tau)^{-1}\tau^* F$ and $\chi(\tau) := \varrho(\tau)\, L(\tau)^{-1}$. Then $\tau^* F = \chi(\tau)\, F\, k(\tau)$ and $\chi(\tau) \in \Lambda_r\mathbf{SU}(2)_\sigma$. □

Corollary: Let $(\xi, \Phi_0, \tilde{z}_0)$ generate the holomorphic frame $\Phi : \widetilde{M} \to \Lambda_r\mathbf{SL}(2,\mathbb{C})_\sigma$ and extended unitary frame $F \in \mathcal{F}(\widetilde{M})$. Let $\tau \in \Delta$ and $\varrho(\tau)$ be a monodromy of Φ and $\chi(\tau)$ a monodromy of F with respect to $\tau \in \Delta$. If $\mathrm{Iso}_r(F) = \{\pm \mathrm{Id}\}$, then $\chi(\tau) = \pm\varrho(\tau)$. □

3.2 A factorization Theorem. In this section we work exclusively with untwisted loops.Consider an analytic map $H : \mathbb{C}^* \to \mathbf{SL}(2,\mathbb{C})$, for which $\mathrm{tr}H$ is not independent of λ. Its eigenvalues are

$$\mu_\pm = \frac{1}{2}\left(\mathrm{tr}H \pm \sqrt{(\mathrm{tr}H)^2 - 4}\right). \tag{3.2.1}$$

We will need to have $\mu_\pm$ holomorphic on some open and dense subset $\mathbb{S}$ of $\mathbb{C}^*$. To this end we slightly generalize the procedure used in [9].

Proposition: Let $g : \mathbb{C}^* \to \mathbb{C}$, $g \not\equiv 0$ be holomorphic. Then there exists some connected, open, and dense subset $\mathbb{S} \subset \mathbb{C}^*$ such that there exists a well defined square root function $\sqrt{g}$ of g on $\mathbb{S}$.

Proof: Since $g(\lambda)$ does not vanish identically it has at most finitely many roots on S^1. Moving slightly inside, if necessary, we can choose any $0 < r \leq 1$ such that g does not vanish on C_r. We form a (scalar) Birkhoff splitting of g on C_r: $g = g_-\lambda^N g_+$. It is easy to verify that, since g is holomorphic on $\mathbb{C}^*$, also g_- and g_+ are holomorphic on $\mathbb{C}^*$. Now we introduce some cuts: Consider the roots of g_+. They are all contained in the complement of I_r. We cut the Riemann sphere from every odd ordered root of g_+ to infinity. Similarly we cut the Riemann sphere from each odd ordered root of g_- to the origin. Thus g_- and g_+ have well defined root functions on the cut domain. If N is odd, then we also need to introduce one further cut from the origin to the point at infinity. This resulting domain will be denoted by $\mathbb{S}$. □

Applying this to the function $g(\lambda) = \mathrm{tr}H^2 - 4$ we obtain

Corollary: The eigenvalue functions $\mu_{\pm}(\lambda)$ of a loop $H \in \Lambda_{\mathbb{C}^*}\mathbf{SL}(2,\mathbb{C})$ are analytic on a connected, open and dense set $\mathbb{S} \subset \mathbb{C}^*$, obtained by making radial branch cuts from odd ordered roots α of $\mathrm{tr}H^2 - 4$ to the origin respectively the point at infinity, depending on whether $|\alpha| < r$ respectively $|\alpha| > r$. In addition there is possibly a cut from 0 to ∞. $\square$

For our approach it will be convenient to diagonalize H

$$Y\,H\,Y^{-1} = \mathrm{diag}[\mu_+,\, \mu_-] \tag{3.2.2}$$

If the matrix entries $H_{1,2}$ and $H_{2,1}$ of H do not vanish identically, then one can choose as diagonalizing matrix Y (see e.g. (3.5.18) in [9])

$$Y = \begin{pmatrix} 1 & -\frac{H_{12}}{u-iv-H_{11}} \\ -\frac{H_{21}}{2iv} & \frac{u+iv-H_{22}}{2iv} \end{pmatrix}, \tag{3.2.3}$$

where H_{ij} denotes the $(i,j)-$coefficient of H and $2u = \mathrm{tr}H$ and $v^2 = u^2 - 1$. If one of the off-diagonal coefficients vanishes identically, then we choose the corresponding matrix Y listed in [9]. In this case the diagonal entries are $\mu_{\pm}$. Note that this works in our setting.(Not only in the somewhat more special setting of [9].)

From the specific form of the matrices Y mentioned above we infer that Y is meromorphic on $\mathbb{S}$, provided the trace $\mathrm{tr}H$ of H is not independent of λ.

The next result is crucial for our characterization of dressing matrices preserving the fundamental group. As before, we denote an open annulus about S^1 by $A_r = \{\lambda \in \mathbb{C} : r < |\lambda| < 1/r\}$ and by $C_r = \{\lambda \in \mathbb{C} : |\lambda| = r\}$ a circle of radius r. With the above notations we have the following

Theorem: Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})$ for some $0 < r \leq 1$ and $H \in \Lambda_{\mathbb{C}^*}\mathbf{SL}(2,\mathbb{C})$ such that the trace of H is not constant. Let $\mathbb{S}$ be as in Corollary 3.2. If $h\,H\,h^{-1}$ is meromorphic on $\mathbb{S} \cap A_s$ for some $0 < s < r$, then h can be factored, on a possibly segmented circle $C_l \cap \mathbb{S}$ for $s \leq l \leq r$, into

$$h = \mathcal{M}\,\mathcal{C}, \tag{3.2.4}$$

where $\mathcal{C}$ is $\mathbf{SL}(2,\mathbb{C})$–valued on $\mathbb{S} \cap C_l$ and $[\,\mathcal{C},\, H\,] = 0$ there. Moreover, $\mathcal{M}$ is $\mathbf{SL}(2,\mathbb{C})$–valued and meromorphic on the connected open set $\mathbb{S} \cap A_s$.

Proof: From the above discussion, there exists some Y such that (3.2.2) holds. Let us write

$$hY^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.2.5}$$

Computing $h\,H\,h^{-1}$ and evaluating the diagonal entries gives that both

$$ad\mu_- - bc\mu_+, \tag{3.2.6}$$

$$ad\mu_+ - bc\mu_- \tag{3.2.7}$$

are meromorphic on $\mathbb{S} \cap A_s$. Multiplying (3.2.6) by μ_+ and (3.2.7) by μ_- and subtracting the resulting equations, implies that $bc(\mu_+^2 - \mu_-^2)$ is meromorphic on $\mathbb{S} \cap A_s$. Multiplying (3.2.6) by μ_- and (3.2.7) by μ_+ and subtracting the resulting equations, implies that $ad(\mu_-^2 - \mu_+^2)$ is meromorphic on $\mathbb{S} \cap A_s$. Assuming $\mu_-^2 - \mu_+^2 = 0$ would imply $\mu_\pm^4 = 1$ and contradict the assumption that $\mathrm{tr}H$ is not independent of λ. Since $\mu_\pm$ are holomorphic on $\mathbb{S} \cap A_s$ both ad and bc are meromorphic on $\mathbb{S} \cap A_s$. Similarly, evaluating the off–diagonal terms of $h\, H\, h^{-1}$ we obtain that both ab and cd are meromorphic on $\mathbb{S} \cap A_s$.

If $d \neq 0$ on some circle C_l, for $s \leq l \leq r$, then we may write $h\, Y^{-1} = \widehat{h}\, \mathrm{diag}[\, 1/d,\, d\,]$ with

$$\widehat{h} := \begin{pmatrix} ad & \frac{b}{d} \\ cd & 1 \end{pmatrix} \text{ meromorphic on } \mathbb{S} \cap A_s. \tag{3.2.8}$$

We define $\mathcal{C} := Y^{-1}\, \mathrm{diag}[\, 1/d, d\,]\, Y$ and $\mathcal{M} := \widehat{h}\, Y$. Then $h = \mathcal{M}\, \mathcal{C}$ with $\mathcal{C} = \mathcal{C}(\lambda)$ defined on $C_l \cap \mathbb{S}$ and $[\, \mathcal{C},\, H\,] = 0$ there. Moreover, $\mathcal{M}$ is meromorphic on $\mathbb{S} \cap A_s$.

If $d \equiv 0$ on an arc $C_l \cap \mathbb{S}$ for some $s \leq l \leq r$, then both $cd \equiv 0$ on this arc and consequently $cd \equiv 0$ in $\mathbb{S} \cap I_r$, and $ad \equiv 0$ on this arc and consequently $ad \equiv 0$ in $\mathbb{S} \cap I_r$. Thus $d \equiv 0$ on $C_l \cap \mathbb{S}$ implies that both $b,\, c \neq 0$ on $\mathbb{S} \cap I_r$ and we may write $h\, Y^{-1} = \widetilde{h}\, \mathrm{diag}[\, c,\, 1/c\,]$ with

$$\widetilde{h} = \begin{pmatrix} a/c & bc \\ 1 & cd \end{pmatrix} \text{ meromorphic on } \mathbb{S} \cap A_s.$$

Set $\mathcal{C} = Y^{-1} \mathrm{diag}[\, c, 1/c\,] Y$ and $\mathcal{M} = \widetilde{h} Y$. Then $h = \mathcal{M}\, \mathcal{C}$ with $[\, \mathcal{C},\, \chi\,] = 0$ and $\mathcal{M}$ meromorphic on $\mathbb{S} \cap A_s$. □

Given an open dense subset $\mathbb{S} \subset \mathbb{C}^*$, obtained by making branch cuts from the points $\{\alpha_j\} \subset \mathbb{C}^*$ to the origin respectively ∞, depending on whether $|\alpha_j| < r$ respectively $|\alpha_j| > r$, let $\hat{\mathbb{S}}$ be the set obtained by making branch cuts according to this rule from the points $\{1/\bar{\alpha}_j\}$. Then the set $\mathbb{S}^* = \mathbb{S} \cap \hat{\mathbb{S}}$ is invariant under $\lambda \mapsto 1/\bar{\lambda}$. Note though that if there are at least two distinct $\alpha_j \in S^1$, then $\mathbb{S}^*$ is not connected.

We now improve on these results by exploiting the reality condition of loops in $\Lambda_{\mathbb{C}^*}\mathbf{SU}(2)$.

Lemma: The eigenvalue functions $\mu_\pm$ of a loop $H \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)$ are of the form $\mu_\pm = u \pm iv$, where $2u = \mathrm{tr}H$ and $v^2 = 1 - u^2 = \lambda^N \cdot \hat{v}^2$ with $N \in 2\mathbb{Z}$ and $\hat{v}$ holomorphic on $\mathbb{S}$. In particular, $S^1 \subset \mathbb{S} = \mathbb{S}^*$.

<u>Proof:</u> A loop $H \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)$ is of the form

$$H(\lambda) = \begin{pmatrix} \dfrac{x(\lambda)}{-y(1/\bar{\lambda})} & \dfrac{y(\lambda)}{x(1/\bar{\lambda})} \end{pmatrix}$$

with analytic functions $x,\, y : \mathbb{C}^* \to \mathbb{C}$. Hence $u|_{S^1} \in [-1,\, 1]$. Consequently, the function v^2 is real and non-negative on S^1 and thus has only even roots on S^1. If

$\{\lambda_1, \ldots, \lambda_m\}$ are the roots of v^2 on S^1, we may write

$$v^2 = \left(\prod_{j=1}^{m} (\lambda - \lambda_j)^{2K_j} \right) s$$

where $s : \mathbb{C}^* \to \mathbb{C}$ is analytic and $s|_{S^1} \neq 0$. We write $s = c_0\, s_-\, \lambda^N\, s_+$ for the scalar Birkhoff decomposition of s on S^1, with $c_0 \in \mathbb{C}^*$ and $N \in \mathbb{Z}$, such that $s_-(\infty) = s_+(0) = 1$. With these normalizations, the reality condition $u^* = u$ implies $s_+^* = s_-$ as well as

$$N = -\sum_{j=1}^{m} K_j \text{ and } \overline{c_0} = c_0 \prod_{j=1}^{m} \lambda_j^{2K_j}.$$

Whence also $s_+^*(\infty) = 1$ and $v^2 = \lambda^N \hat{v}^2$ with

$$\hat{v}^2 = c_0\, s_-\, s_+ \prod_{j=1}^{m} (\lambda - \lambda_j)^{2K_j}. \tag{3.2.9}$$

Since s is analytic in $\mathbb{C}^*$, both $s_\pm$ are analytic in $\mathbb{C}^*$, in fact $s_+(\lambda)$ is entire, and thus also $\hat{v}^2$ is analytic in $\mathbb{C}^*$. There are two cases depending on the parity of $N \in \mathbb{Z}$. If N is even, then making branch-cuts from the odd ordered roots of s_+ to the point at infinity and branch-cuts from the odd ordered roots of s_- to the origin gives an open and dense set $\mathbb{S} \subset \mathbb{C}^*$ on which $\hat{v}^2$ has an analytic square root. Since the roots of $\hat{v}^2$ on S^1 are all even, we have that also v^2 has an analytic square root on $\mathbb{S}$ and $S^1 \subset \mathbb{S}$. Further, $s_+^* = s_-$ implies $\mathbb{S}^* = \mathbb{S}$.

Assume that $N \in \mathbb{Z}$ is odd. Then $\mathbb{S}$ is the set obtained as above but with an additional branch cut from the origin to the point at infinity. Now $u = \sqrt{\mathrm{tr}^2 H - 4}$ has an absolutely convergent power series expansion on S^1 and is thus well defined, so there can not be a branch cut through S^1. Hence N is even. $\square$

Corollary: Let $h \in \Lambda_r^+ \mathbf{SL}(2, \mathbb{C})$ for some $0 < r \leq 1$ and $H \in \Lambda_{\mathbb{C}^*} \mathbf{SU}(2)$ with non-constant trace function and let $\mathbb{S}$ be as in Lemma 3.2. If $h\, H\, h^{-1}$ is meromorphic on $\mathbb{S} \cap A_s$ for some $0 < s < r$ and in addition we require that $(h\, H\, h^{-1})^* = (h\, H\, h^{-1})^{-1}$ for all $\lambda \in \mathbb{S} \cap A_s$, then

$$h = U\, C \tag{3.2.10}$$

with meromorphic loops U, C on a cut but connected annulus respectively a possibly segmented circle C_l for some $s < l < r$ with $[\, C,\, H\,] = 0$ there, and $U \in \Lambda_{r'} \mathbf{SU}(2)$ for some $r' \in (0, 1)$.

Proof: Note that we have $S^1 \subset \mathbb{S} = \mathbb{S}^*$ under our assumptions. We apply Theorem 3.2 and decompose $h = \mathcal{M}\, \mathcal{C}$ on C_l for some $s < l < r$ with $\mathcal{M}$ meromorphic on $\mathbb{S} \cap A_s$ and $[\, \mathcal{C},\, H] = 0$. Consequently, we have $h\, H\, h^{-1} = \mathcal{M}\, H\, \mathcal{M}^{-1}$ on $\mathbb{S} \cap A_s$ and $(h\, H\, h^{-1})^* = (h\, H\, h^{-1})^{-1}$ is equivalent to

$$[\, \mathcal{M}^*\, \mathcal{M},\, H\,] = 0 \tag{3.2.11}$$

on $\mathbb{S} \cap A_s$. Next, we seek a loop L, meromorphic on a subset of $\mathbb{S} \cap A_s$, of the form $L = f\,\mathrm{Id} + g\,\mathcal{M}^*\mathcal{M}$ with meromorphic functions f, g on $\mathbb{S} \cap A_s$, such that

$$(\mathcal{M}L^{-1})^* = (\mathcal{M}L^{-1})^{-1}. \tag{3.2.12}$$

Then $h = \mathcal{M}L^{-1}L\mathcal{C}$ is the desired factorization. Equation (3.2.12) is equivalent to $\mathcal{M}^*\mathcal{M} = L^*L$ and by setting $P := \mathcal{M}^*\mathcal{M}$, yields

$$\begin{aligned} P &= f f^*\mathrm{Id} + (f^*g + f\,g^*)P + g\,g^*P^2 \\ &= (ff^* - gg^*)\mathrm{Id} + (f^*g + fg^* + gg^*\mathrm{tr}(P))P \end{aligned} \tag{3.2.13}$$

since $P^2 = \mathrm{tr}(P)P - \mathrm{Id}$ by Cayley-Hamilton. The Ansatz $f = f^* = g$ reduces (3.2.13) to

$$g^2 = \frac{1}{2 + \mathrm{tr}P}. \tag{3.2.14}$$

If we denote the entries of $\mathcal{M}$ by m_{ij}, then the function

$$\mathrm{tr}P(\lambda) = \sum_{i,j=1}^{2} m_{ij}(\lambda)m_{ij}^*(\lambda) \tag{3.2.15}$$

is meromorphic on $\mathbb{S} \cap A_s$ and is real and positive on $S^1 \subset \mathbb{S} \cap A_s$. With the help of a scalar Birkhoff decomposition of g^2 it is straightforward to see that there exists a well defined square root of (3.2.14) on S^1, which extends meromorphically to the set $\mathbb{S}_P \subset \mathbb{S}$, obtained by making branch-cuts from the odd ordered roots of $2 + \mathrm{tr}P$ in the usual fashion. Note that $S^1 \subset \mathbb{S}_P$ and $\mathbb{S}_P^* = \mathbb{S}_P$. Now that $g = (2 + \mathrm{tr}P)^{-1/2}$ is meromorphic on $\mathbb{S}_P \cap A_s$, we can set $U := \mathcal{M}L^{-1}$, which is meromorphic on $\mathbb{S}_P \cap A_s$ with $U^* = U^{-1}$, and $C := L\mathcal{C}$, which is defined on $C_l \cap \mathbb{S}_P$ with $[\,C, H\,] = 0$. It remains to show that there exists an $r' \in (0,\,1)$ such that $U \in \Lambda_{r'}\,\mathbf{SU}(2)$.

We choose $0 < r' < 1$ such that both of the following conditions hold:

(i) The only poles of $\mathcal{M}$ in $A_{r'} \cap \mathbb{S}$ are on S^1,

(ii) $A_{r'} \cap \mathbb{S}_P = A_{r'}$.

Denote the pole set of $\mathcal{M}$ in $A_{r'}$ by $\mathcal{P} = \{p_1, \ldots, p_K\} \subset S^1$. Let $V_j \subset A_{r'}$ be an open neighborhood of p_j such that $V_j \cap \mathcal{P} = \{p_j\}$. Let $V_j^* = V_j \setminus \{p_j\}$. Since U is unitary on $\mathbb{S}_P \cap U_j^* \cap S^1$, its entries are bounded. Hence U extends analytically to p_j for all $j = 1, \ldots, K$. Consequently, U is analytic in $A_{r'}$ with $U^* = U^{-1}$. Thus $U \in \Lambda_{r'}\,\mathbf{SU}(2)$, concluding this proof. □

In the following sections we will apply the results of the present section to the twisted loop groups.

3.3 Let $F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ and $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ and let $\hat{F} = h\#F$. Then $\hat{F} \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$. Let $\tau \in \Delta$ and $\chi(\tau)$, $\hat{\chi}(\tau)$ be monodromies of F respectively $\hat{F}$. Writing $hF = \hat{F}B$ for an Iwasawa decomposition and $\tau^*F = \chi(\tau)\,F\,k(\tau)$ and $\tau^*\hat{F} = \hat{\chi}(\tau)\,\hat{F}\,\hat{k}(\tau)$, we obtain

$$\hat{\chi}(\tau)^{-1}h\,\chi(\tau)\,h^{-1}\hat{F} = \hat{F}\,\hat{k}(\tau)\,(\tau^*B)\,k(\tau)^{-1}B^{-1}. \tag{3.3.1}$$

Since $\hat{k}(\tau)\,(\tau^* B)\,k(\tau)^{-1}B^{-1} \in \mathcal{G}_r(\widetilde{M})$, this proves that

$$\hat{\chi}(\tau)^{-1} h\,\chi(\tau)\,h^{-1} \in \mathrm{Iso}_r(\hat{F}). \tag{3.3.2}$$

Since $F(\tilde{z}_0) = \hat{F}(\tilde{z}_0) = \mathrm{Id}$, evaluation of $hF = \hat{F}B$ at the base-point $\tilde{z}_0$ yields $h = B(\tilde{z}_0)$. Evaluating equation (3.3.1) at $\tilde{z}_0$ and solving for $\hat{\chi}(\tau)$ gives

$$\begin{aligned} \hat{\chi}(\tau) &= h\,\chi(\tau)\,k(\tau,\tilde{z}_0)\,B\,(\tau(\tilde{z}_0))^{-1}\,\hat{k}(\tau,\tilde{z}_0)^{-1} \\ &= h\,\chi(\tau)\,h^{-1}\,L,\; L \in \mathrm{Iso}_r(\hat{F}) \\ &= h\,\chi(\tau)\,b\,h^{-1},\; b := h^{-1}L\,h \in \mathrm{Iso}_r(F). \end{aligned} \tag{3.3.3}$$

This leads us to an application of Theorem 3.2 and shows that if a dressing matrix preserves the topology, then it factorizes as in (3.2.4).

We now denote by $\mathbb{S} = \mathbb{S}^*$ the set obtained from the eigenvalues of a monodromy χ.

Theorem: Let $\tau \in \Delta$ and $F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ with monodromy $\chi = \chi(\tau)$. We assume that the trace of χ is not constant. Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ such that $h\,\chi\,h^{-1}$ is meromorphic on $\mathbb{S} \cap A_s$ for some $0 < s < r$, where $\mathbb{S}$ is as in Lemma 3.2. If $\hat{F} = h\#F$ has a monodromy $\hat{\chi}$ with respect to τ, and if $\hat{\mathbb{S}}$ is defined for $\hat{\chi}$ according to Lemma 3.2, then h can be factored, on a possibly segmented circle $C_l \cap \mathbb{S} \cap \hat{\mathbb{S}}$ for $l \in [s,\, r]$, into $h = \mathcal{M}\,\mathcal{C}$ where $\mathcal{C} = \mathcal{C}(\tau,\,\lambda)$ is twisted $\mathbf{SL}(2,\mathbb{C})$–valued and defined on $C_l \cap \mathbb{S} \cap \hat{\mathbb{S}}$ and $[\,\mathcal{C},\,\chi\,] = 0$ there, and $\mathcal{M} = \mathcal{M}(\tau,\,\lambda)$ is twisted $\mathbf{SU}(2)$–valued and meromorphic on the connected open set $S^1 \subset \mathbb{S} \cap \hat{\mathbb{S}} \cap A_s$.

Proof: By the above (3.3.3), for the dressed monodromy we have $\hat{\chi} = h\,\chi\,h^{-1}\,\hat{L}$ with $\hat{L} \in \mathrm{Iso}_r(\hat{F})$. Clearly, $\hat{L}$ is defined and meromorphic on $\mathbb{S} \cap A_s$. Moreover, we can write $\hat{\chi} = h\,\chi\,Lh^{-1}$, where $L \in \mathrm{Iso}_r(F)$. From this we conclude that the eigenvalue functions of $\chi\,L$ are the same as those of $\hat{\chi}$. At this point we undress every occurring matrix and continue, until the very end of the proof, to work in the undressed setting. We note that the domains of definition of the dressed and the undressed matrices are in a natural correspondence and that unitary matrices are mapped to unitary matrices, while the "positive matrices" almost correspond. For simplicity of notation we will use the same notation as in the twisted situation. First we note that we can apply Theorem 3.2 to $h\,\chi\,h^{-1}$. Hence $h = \mathcal{M}\,\mathcal{C}$, where $\mathcal{M}$ has values in $\mathbf{SL}(2,\mathbb{C})$ and is defined on $\mathbb{S}$. As a consequence, $\hat{\chi} = \mathcal{M}\chi\mathcal{M}^{-1}\hat{L} = \mathcal{M}\chi\,L'\mathcal{M}^{-1}$. From this expression we infer that L' is defined and meromorphic on $\mathbb{S}$. Now we can apply almost verbatim the proof of 3.2, if we replace everywhere $\mathbb{S}$ by $\mathbb{S} \cap \hat{\mathbb{S}} = \tilde{\mathbb{S}}$ and obtain that we can assume that $\mathcal{M}$ is defined and holomorphic on some open neighborhood of S^1 and unitary on S^1. Finally, twisting every occurring matrix again we obtain the desired result. $\square$

When the associated family of $F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ possesses umbilic points, then, since dressing preserves umbilic points [29], $\mathrm{Iso_r(\hat{F})} = \{\pm\mathrm{Id}\}$ for $\hat{F} = h\#F$ and (3.3.2) implies

$$\hat{\chi}(\tau) = \pm h\,\chi(\tau)\,h^{-1} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma. \tag{3.3.4}$$

This allows us to restate Corollary 3.2 for the trivial isotropy case:

Corollary: Let $F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ with $\mathrm{Iso}(F) = \{\pm\mathrm{Id}\}$. Let $\chi = \chi(\tau) \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ be a monodromy of F with non-constant trace. Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ for some $r \in (0,\, 1]$. If $h\#F$ also has a monodromy with respect to τ, then h can be factored into $h = U\,C$ with $[\,C,\, \chi\,] = 0$ and $U = U(\tau,\, \lambda)$ the meromorphic extension of an element of $\Lambda_{r'}\mathbf{SU}(2)_\sigma$ for some $r' \in [r,\, 1]$. □

We were not able to prove the corresponding result for a general $F \in \mathcal{F}_{\mathrm{Id}}(\widetilde{M})$ without the isotropy condition. It turns out, that we do have an analogous result for the dressing orbit of the vacuum, to which we now turn our attention.

4 Dressing the vacuum

4.1 Definitions. The standard round cylinder (="the vacuum") has the conformal structure of the punctured plane $\mathbb{C}^*$. If we identify $\mathbb{C}^* \cong \mathbb{C}/q\,\mathbb{Z}$ for some $q \in \mathbb{C}^*$, then

$$\exp : \mathbb{C} \to \mathbb{C}^*,\; w \mapsto z = \exp(q\,w) \tag{4.1.1}$$

is the universal covering map. The group of deck transformations $\Delta \cong \mathbb{Z}$ is generated by the translation

$$\tau_q : w \mapsto w + q. \tag{4.1.2}$$

An extended unitary frame of the associated family of the vacuum is given by

$$F_c(w,\, \lambda) = \exp((w\lambda^{-1} - \overline{w}\lambda)\, A) \tag{4.1.3}$$

where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.1.4}$$

A monodromy with respect to τ_q of F_c is given by

$$\chi_c(\tau_q) = \exp((q\lambda^{-1} - \bar{q}\lambda)\, A). \tag{4.1.5}$$

The map $F_c : \mathbb{C} \to \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ is obtained in the DPW framework from the triple $((\lambda^{-1} + \lambda)A\, dw,\, \mathrm{Id},\, 0)$. Note that we have chosen the base-point $w_0 = 0$. The corresponding holomorphic frame is

$$\Phi_c(w,\, \lambda) = \exp((\lambda^{-1} + \lambda)\, w\, A) \tag{4.1.6}$$

with monodromy with respect to τ_q given by

$$\varrho_c(\tau_q) = \exp((\lambda^{-1} + \lambda)\, q\, A). \tag{4.1.7}$$

Note that if $q \in i\mathbb{R}$, then $\varrho_c(\tau_q) \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ and $\varrho_c(\tau_q) = \chi_c(\tau_q)$.

4.2 Commuting flows. Denote the abelian sub-algebra of elements of $\Lambda_{\mathbb{C}^*}\mathbf{sl}(2,\mathbb{C})_\sigma$ that commute with A and have a pole at $\lambda = 0$ by

$$\mathcal{Z} = \left\{ \phi\, A : \phi(\lambda) = \Sigma \phi_j \lambda^j \text{ odd },\ j \geq -K,\ K = 2g+1 \in \mathbb{N} \right\}.$$

Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$, $\eta \in \mathcal{Z}$ and define $h\sharp\eta$ via the r–Iwasawa decomposition

$$h \exp(\eta) = U\, h\sharp\eta. \tag{4.2.1}$$

Further, for $F = h\#F_c$ define $F\sharp\eta := (h\sharp\eta)\#F_c$. By [5], Proposition 4.1, this is an action on the dressing orbit of F_c: If $\eta_1,\, \eta_2 \in \mathcal{Z}$, then $F\sharp(\eta_1 + \eta_2) = (F\sharp\eta_2)\sharp\eta_1 = (F\sharp\eta_1)\sharp\eta_2$ and if $\eta_1 - \eta_2 \in \Lambda_r^+\mathbf{sl}(2,\mathbb{C})$, then $F\sharp\eta_1 = F\sharp\eta_2$. By construction $\zeta = \phi A \in \mathcal{Z}$ is analytic for $\lambda \in \mathbb{C}^*$ with a pole at $\lambda = 0$. Expanding $\phi(\lambda) = \sum_{j\geq -K} \phi_j \lambda^j$, observe that $\exp(\zeta)\#F_c = \exp(\eta)\#F_c$ where $\eta = \phi' A$ and $\phi' = \sum_{j=-K}^{0} \phi_j \lambda^j$ is analytic on $\mathbb{CP}^1 \setminus \{0\}$. Since $\eta^* = \phi^* A$ is analytic in $\mathbb{C}$, $e^{\eta - \eta^*} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ and $e^\zeta \# F_c = e^{\eta-\eta^*} F_c$. Hence, when dressing with e^ζ, we restrict without loss of generality to $\zeta \in \mathcal{Z}$ of the form

$$\zeta = \sum_{j=1}^{K} (\phi_j \lambda^{-j} - \bar{\phi}_j \lambda^j) A \tag{4.2.2}$$

where $\phi_j \in \mathbb{C}$ and summation is over odd indices $j = 1, 3, \ldots, K = 2g+1 \in \mathbb{N}$. Then

$$\exp(\zeta) \,\#\, F_c(w,\, \lambda) = \exp\Big(\sum_{j=3}^{K} (\phi_j \lambda^{-j} - \bar{\phi}_j \lambda^j) A \Big)\, F_c(w + \phi_1,\, \lambda). \tag{4.2.3}$$

For $\zeta \in \mathcal{Z}$ of the form (4.2.2) we have $e^{t\zeta} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ for all $t \in \mathbb{R}$. Hence, for any extended unitary frame $F \in \mathcal{F}(\widetilde{M})$, we have $e^{t\zeta}\#F = e^{t\zeta}F$ which on the level of the immersions has the effect

$$f \longmapsto \exp(t\,\zeta)\, f\, \exp(-t\,\zeta) - \tfrac{1}{2H} i\lambda t \tfrac{\partial\zeta}{\partial\lambda}.$$

Each $\zeta \in \mathcal{Z}$ defines a flow on the dressing orbit of the standard round cylinder by

$$F_t := F\sharp t\zeta = (h\sharp t\zeta)\#F_c = U(\lambda,\, t)^{-1} h \# e^{t\zeta} F_c \tag{4.2.4}$$

where we have defined $U(\lambda,\, t)$ via the r–Iwasawa decomposition $he^{t\zeta} = U(\lambda,\, t)(h\sharp t\zeta)$ and $F = h\#F_c$ for some $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. The flow F_t is called *trivial* if and only if $F_t = V(t)FV(t)^{-1}$ for a λ–independent map $V : \mathbb{R} \to \mathbf{SU}(2)$ and F is of *finite type* if and only if the subspace $\mathcal{Z}' \subset \mathcal{Z}$ of trivial flows has finite co-dimension in $\mathcal{Z}$.

4.3 Monodromy. Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ and $hF_c = FB$ the r–Iwasawa decomposition of hF_c such that at the base-point $w_0 = 0$ we have $F_c(0,\, \lambda) = F(0,\, \lambda) = \mathrm{Id}$ and $h(\lambda) = B(0,\, \lambda)$. Let us assume that F has symmetry $(\tau_q,\, \chi(\tau_q))$. From (3.3.3) we obtain

$$\chi(\tau_q) = h\, \chi_c(\tau_q)\, B(q)^{-1}. \tag{4.3.1}$$

For completeness of exposition we prove the following result, first obtained in sections 3.1–3.4 of [6].

Lemma: Let $h \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ and F_c be as in (4.1.3). Assume that $h\#F_c$ has monodromy $\chi(\tau_q) \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$. Then there exists an odd function $f_q : C_r \to \mathbb{C}$ with a holomorphic extension to I_r such that $B(q)^{-1} = \exp(f_q A)\, h^{-1}$. Setting

$$p_q(\lambda) = q\lambda^{-1} - \bar{q}\lambda + f_q(\lambda), \tag{4.3.2}$$

we may write the monodromy of $h\#F_c$ as

$$\chi(\tau_q) = h\, \exp(p_q A)\, h^{-1}. \tag{4.3.3}$$

Proof: Let $\chi_c(\tau_q)$ be the monodromy of F_c as in (4.1.5). Then

$$\chi_c(\tau_q) F_c = \tau_q^* F_c = h^{-1}\chi(\tau_q) h F_c B^{-1} \tau_q^* B \tag{4.3.4}$$

in combination with (4.3.1) gives $B(q)^{-1}B(0)F_c = F_c\, \tau_q^* B^{-1} B$.

Hence $[B(q)^{-1}B(0), A] = 0$ and for the matrix

$$T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \tag{4.3.5}$$

we have $TAT^{-1} = \sigma_3$ and $[TB(q)^{-1}B(0)T^{-1}, \sigma_3] = 0$. Thus $TB(q)^{-1}B(0)T^{-1} = \mathrm{diag}[s_q,\, s_q^{-1}]$ for some holomorphic function $s_q : I_r \to \mathbb{C}^*$. Defining $s_q = \exp(f_q)$ yields $B(q)^{-1}B(0) = \exp(f_q A)$. Since $B(0) = h$, this proves the claim. $\square$

The point of Lemma 4.3 is that we can now write a monodromy of an extended frame in the dressing orbit of the vacuum as

$$\cosh(p_q)\, \mathrm{Id} + \sinh(p_q)\, h\, A\, h^{-1} \tag{4.3.6}$$

and much of the analysis reduces to the study of the two scalar functions

$$\alpha = \cosh(p_q) \text{ and } \beta = \sinh(p_q). \tag{4.3.7}$$

In analogy to Corollary 3.3 we have the following result for the monodromy representation of surfaces in the dressing orbit of the cylinder.

Theorem: Let $h \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ and F_c be as in (4.1.3). Assume that $h\#F_c$ has symmetry $(\tau_q,\, \chi(\tau_q))$. Then $h = \mathcal{M}(\tau_q)\, \mathcal{C}(\tau_q)$ with $\mathcal{M}(\tau_q) \in \Lambda_{r'}\mathbf{SU}(2)_\sigma$ for suitable $r \le r' < 1$ and $[\, \mathcal{C}(\tau_q),\, A\,] = 0$.

Proof: For T defined in (4.3.5) and by Lemma 4.3, we may write

$$\chi(\tau_q) = h\, T^{-1}\, \mathrm{diag}[\, \exp(-p_q),\, \exp(p_q)\,]\, T\, h^{-1}. \tag{4.3.8}$$

for $0 <\mid \lambda \mid< r$. Let $hT^{-1} = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The diagonal entries of $\chi(\tau_q)$ are analytic functions in $\lambda \in \mathbb{C}^*$ and given by

$$\begin{aligned} &a\,d\,\exp(-p_q) - b\,c\,\exp(p_q), \\ &a\,d\,\exp(p_q) - b\,c\,\exp(-p_q). \end{aligned} \tag{4.3.9}$$

Adding these implies that $\alpha = \cosh(p_q)$ is an analytic function on $\mathbb{C}^*$. More precisely, α has an analytic extension from $0 <\mid \lambda \mid< r$ to $\mathbb{C}^*$. Solving for $\exp(p_q)$ we obtain $\exp(p_q) = \alpha + \sqrt{\alpha^2 - 1}$. This is analytic on the set $\mathbb{S} \subset \mathbb{C}^*$ obtained by making branchcuts from the odd–ordered roots of $\alpha^2 - 1$ in the usual fashion. Note that the equation (4.3.8) above implies that $\mu_\pm = \exp(\pm p_q)$ are the eigenvalues of $\chi(\tau_q)$. Hence, by Lemma 3.2 we know that $\mathbb{S}$ is open, connected, and dense in $\mathbb{C}^*$. Moreover we have $S^1 \subset \mathbb{S}$ and $\mathbb{S} = \mathbb{S}^*$.

Consequently, $\exp(-p_q)$ is holomorphic on $\mathbb{S}$. Multiplying the $(1,1)$ entry of $\chi(\tau_q)$ by $\exp(p_q)$ and the $(2,2)$ entry of $\chi(\tau_q)$ by $\exp(-p_q)$ and subtracting the resulting equations implies that $bc\sinh(p_q)$ is holomorphic on $\mathbb{S}$. Note that $\beta = \sinh(p_q) \not\equiv 0$. Hence bc is meromorphic on $\mathbb{S}$. Similarly one shows that ad is meromorphic on $\mathbb{S}$.

The off–diagonal terms of $\chi(\tau_q)$ are $2ab\beta$ and $-2cd\beta$. Hence both ab and cd are meromorphic on $\mathbb{S}$. If $d \neq 0$ on the circle C_r, then as in (3.2.8), we may write $hT^{-1} = \hat{h}D_1$ with $D_1 = \mathrm{diag}[1/d,\, d]$ and $\hat{h}$ meromorphic on $\mathbb{S}$.

Defining $\mathcal{C}(\tau_q) = T^{-1}D_1T$, $\mathcal{M}(\tau_q) = \hat{h}T$ yields $h = \mathcal{M}(\tau_q)\mathcal{C}(\tau_q)$ with $[\,\mathcal{C}(\tau_q),\, A\,] = 0$ and $\mathcal{M}(\tau_q)$ meromorphic on $\mathbb{S}$. Arguing as in the proof of Theorem 3.2, $d \equiv 0$ on an arc $C_s \cap \mathbb{S}$ implies that both $b,\, c \neq 0$ on $\mathbb{S}$ and we may write $hT^{-1} = \widetilde{h}D_2$ with $D_2 = \mathrm{diag}[c,\, 1/c]$ and $\widetilde{h}$ meromorphic on $\mathbb{S}$. Defining $\mathcal{C}(\tau_q) = T^{-1}D_2T$, $\mathcal{M}(\tau_q) = \widetilde{h}T$ yields $h = \mathcal{M}(\tau_q)\mathcal{C}(\tau_q)$ with $[\,\mathcal{C}(\tau_q),\, A\,] = 0$ and $\mathcal{M}(\tau_q)$ meromorphic on $\mathbb{S}$ in this case. In either case, on C_r we may write

$$\chi(\tau_q) = \mathcal{M}(\tau_q)\,\exp(p_q\,A)\,\mathcal{M}(\tau_q)^{-1}. \tag{4.3.10}$$

Introducing an additional cut from 0 to $-\infty$ we obtain a simply connected domain $\hat{\mathbb{S}} = \hat{\mathbb{S}}^*$, on which we can take the logarithm of the holomorphic function $\exp(p_q)$, thus defining p_q on $\hat{\mathbb{S}}$. Then on $\hat{\mathbb{S}}$ we combine $\chi(\tau_q)^* = \chi(\tau_q)^{-1}$ with (4.3.10) to obtain

$$\mathcal{M}^*(\tau_q)\,\mathcal{M}(\tau_q)\,\exp(-p_q\,A) = \exp(p_q^*\,A)\,\mathcal{M}^*(\tau_q)\,\mathcal{M}(\tau_q). \tag{4.3.11}$$

For T defined in (4.3.5), recall that $T\,A\,T^{-1} = \mathrm{diag}[1,\, -1]$. Since $T \in \mathbf{SU}(2)$, the matrix $X := T\,\mathcal{M}^*(\tau_q)\,\mathcal{M}(\tau_q)\,T^{-1}$ is hermitian and of the form $X = W\,W^*$. In particular, if we write $X = \left(\begin{smallmatrix} x & y \\ \bar{y} & z \end{smallmatrix}\right)$, then $x \neq 0$. We can rewrite (4.3.11) as $X\,T\,\exp(-p_q\,A)\,T^{-1} = T\,\exp(p_q^*\,A)\,T^{-1}\,X$ and obtain

$$\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix}\begin{pmatrix} \exp(-p_q) & 0 \\ 0 & \exp(p_q) \end{pmatrix} = \begin{pmatrix} \exp(p_q^*) & 0 \\ 0 & \exp(-p_q^*) \end{pmatrix}\begin{pmatrix} x & y \\ \bar{y} & z \end{pmatrix}. \tag{4.3.12}$$

Since $x \neq 0$, then $\exp(-p_q) = \exp(p_q^*)$ on $\hat{\mathbb{S}}$ and we conclude that $\cosh(p_q^*) = \cosh(p_q)$ and $\sinh(p_q^*) = -\sinh(p_q)$. Consequently $\exp(p_q^*\,A) = \exp(-p_q\,A)$ on $\hat{\mathbb{S}}$ and equation

(4.3.11) reads

$$[\,\mathcal{M}^*(\tau_q)\mathcal{M}(\tau_q),\, \exp(-p_q\, A)\,] = 0. \tag{4.3.13}$$

From the proof of Corollary 3.2 we now obtain some $L(\tau_q)$ and $r' \in [r, 1)$ such that $\mathcal{M}(\tau_q)L(\tau_q) \in \Lambda_{r'}\mathbf{SU}(2)_\sigma$ and $[\,L^{-1}(\tau_q),\, \mathcal{C}(\tau_q)\,] = 0$, and concludes the proof. □

Before going on to evaluate the result above to examples we would like to relate the work above to [6]. Of particular interest to us is equation (3.6.7) there:

$$h_+ = \frac{i}{2\sqrt{\tilde{b}}}\begin{pmatrix} (x - x^{-1})\tilde{b} & (x + x^{-1})\tilde{b} \\ (x + x^{-1}) - (x - x^{-1})\tilde{a} & (x - x^{-1}) - (x + x^{-1})\tilde{a} \end{pmatrix} \tag{4.3.14}$$

where $\tilde{a}$ and $\tilde{b}$ denote the entries of the matrix hAh^{-1} in the (11)–position and the (12)–position respectively. This is the general form of a dressing matrix having on χ the same effect as h. The question addressed in the Theorem above is thus equivalent to the question whether one can choose x so that h_+ is unitary on S^1.

Lemma: The matrix h_+ is unitary on S^1 if and only if

$$|\, x\,|^2 = \frac{1 + \tilde{a} + \sqrt{|\tilde{b}|^2}}{1 - \tilde{a} + \sqrt{|\tilde{b}|^2}}. \tag{4.3.15}$$

This equation can be solved with some x holomorphic at $\lambda = 0$ and well defined on some sufficiently well cut complex plane.

Proof: The unitarity condition for h_+ yields two equations. However, a straightforward computation shows that these two equations are equivalent. It thus remains to consider

$$(x^* - (x^*)^{-1})\sqrt{\tilde{b}^*} = \frac{1}{\sqrt{\tilde{b}}}[(x - x^{-1}) - (x + x^{-1})\tilde{a}]. \tag{4.3.16}$$

Splitting this equation into real and imaginary part yields two equations. However, again, a straightforward computation shows that these two equations are equivalent. It thus suffices to consider

$$(Re(x)^* - (Re(x)^*)^{-1})\sqrt{|\tilde{b}|^2} = (Re(x) - Re(x)^{-1}) - (Re(x) + Re(x)^{-1})\tilde{a}. \tag{4.3.17}$$

Note that we have used here that $\tilde{a}$ is real on S^1 by [6], Theorem 3.7. Using $x^{-1} = x/|\, x\,|^2$ this equation rewrites directly into the claim. □

Corollary: h_+ can be chosen diagonal if and only if $\tilde{a} = 0$. In this case the diagonal entries of h_+ are unitary on S^1.

Proof: Using (4.3.14) for general x we see that h_+ is diagonal only if $x + x^{-1}$ vanishes identically, since $\tilde{b}$ cannot vanish by b) and c) of [6], Theorem 3.7. Inserting this into the second off-diagonal expression for h_+ we obtain $\tilde{a} = 0$. The converse is obvious

choosing $x = i$. Assume now that h_+ has been chosen as a diagonal matrix. Since $\tilde{a}$ vanishes in this case as just shown, c) and e) of [6] imply that $\tilde{b}$ is unitary on S^1. Finally, in view of (4.3.14), we obtain

$$h_+ = \frac{i}{2\sqrt{\tilde{b}}}(x - x^{-1})\mathrm{diag}[b,\, 1]. \tag{4.3.18}$$

Moreover, $x = -x^{-1}$, whence $x = \pm i$. The condition $\det h_+ = 1$ is now satisfied. Thus necessarily $h_+ = \pm\mathrm{diag}[\sqrt{\tilde{b}},\, 1/\sqrt{\tilde{b}}]$ and the diagonal entries of h_+ are unitary on S^1. $\square$

An immediate consequence of the above discussion leads to following observation.

Corollary: Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ such that $h\, \exp(p_q A)\, h^{-1} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ for some translation τ_q. If h is diagonal, then h^4 is rational on $\mathbb{CP}^1$ and h^2 is analytic in $\mathbb{CP}^1 \backslash \mathcal{H}$, where $\mathcal{H}$ is the set obtained by making appropriate branchcuts from the points lying in the set

$$\mathcal{H}' = \{\lambda \in \mathbb{CP}^1 : |\lambda| > r \text{ and } p_q(\lambda) \in \pi i\mathbb{Z}\} \cup \{\text{ singularities of } p_q(\lambda)\} \tag{4.3.19}$$

and h is unitary on S^1. $\square$

Remark: Writing $h = \mathrm{diag}[a,\, 1/a]$, we have that $\chi(\tau_q) = \alpha\,\mathrm{Id} + \beta\,\mathrm{off}[a^2,\, a^{-2}]$ is analytic on $\mathbb{C}^*$. Hence a^2 and a^{-2} are analytic in $\mathcal{H}$ and even in λ. Following the procedure for splitting a matrix we first untwist h. The above procedure for constructing the factorisation $h = \mathcal{M}\mathcal{C}$ gives

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} 1 + a^2 & 1 - a^2 \\ -1 + a^{-2} & 1 + a^{-2} \end{pmatrix}, \mathcal{C} = \frac{1}{2}\begin{pmatrix} a + 1/a & a - 1/a \\ a - 1/a & a + 1/a \end{pmatrix}, \tag{4.3.20}$$

where a really denotes the untwisted function. Now we need to twist again. This way we obtain the original function a and in addition the off-diagonal terms are multiplied by λ (in the $(12-)$position) and by λ^{-1} in the $(21)-$position respectively. Clearly this leads to a matrix $\mathcal{M}$ which is twisted as required.

4.4 Delaunay Surfaces. As an example, we discuss how to dress the vacuum into a Delaunay surface and compute the splitting of this dressing matrix according to Theorem 4.3. It is proven in [14] that an associated family of the Delaunay surface with neck radius ω is generated by the triple $(D\, dz/z,\, \mathrm{Id},\, 0)$, where

$$D = \begin{pmatrix} 0 & \alpha\lambda^{-1} + \beta\lambda \\ \beta\lambda^{-1} + \alpha\lambda & 0 \end{pmatrix}, \tag{4.4.1}$$

and $\alpha,\, \beta \in \mathbb{R}$ with $\alpha + \beta = 1/2$ and $\omega = \frac{1}{2H}(1 - \sqrt{1 - 16\alpha\beta})$. The monodromy of the unitary frame with respect to the translation τ_q for $q = 2\pi i$ is given by $\exp(2\pi i D)$.

Therefore, to dress the standard cylinder into a Delaunay surface, it suffices to determine a diagonal $h = \mathrm{diag}[a,\, 1/a] \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ for some suitable $r > 0$ such that $p_q h A h^{-1} = D$, which is equivalent to the two equations

$$a^{\pm 2}(q\lambda^{-1} - \bar{q}\lambda + f_q) = 2\pi i(\alpha\lambda^{\mp 1} + \beta\lambda^{\pm 1}).$$

Solving both equations for a^2 and equating gives

$$2\pi i\sqrt{(\alpha + \beta\lambda^2)(\beta + \alpha\lambda^2)} = q - \bar{q}\lambda^2 + \lambda f_q, \tag{4.4.2}$$

which in turn yields $a^2 = \sqrt{(\alpha + \beta\lambda^2)/(\beta + \alpha\lambda^2)}$. Evaluating (4.4.2) at $\lambda = 0$ gives $q = 2\pi i\sqrt{\alpha\,\beta}$. If $\rho := \min\{|\sqrt{\alpha/\beta}|,\, |\sqrt{\beta/\alpha}|\}$, then $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ for $0 < r < \rho$. It is easily verified that $p(\lambda) = 2\pi i\sqrt{-\det D}$ vanishes at the singularities of h. In summary, the matrix that dresses the vacuum into a Delaunay surface with neck radius ω is given by

$$h(\lambda) = \mathrm{diag}[\,\sqrt[4]{\tfrac{\alpha+\beta\lambda^2}{\beta+\alpha\lambda^2}},\ \sqrt[4]{\tfrac{\beta+\alpha\lambda^2}{\alpha+\beta\lambda^2}}\,] \tag{4.4.3}$$

Untwisting h, factoring $h = \mathcal{M}\mathcal{C}$ as in (4.3.20) and twisting back gives

$$\mathcal{M} = \frac{1}{2}\begin{pmatrix} 1 + \sqrt{\frac{\alpha+\beta\lambda^2}{\beta+\alpha\lambda^2}} & \lambda - \lambda\sqrt{\frac{\alpha+\beta\lambda^2}{\beta+\alpha\lambda^2}} \\ \lambda^{-1}\sqrt{\frac{\beta+\alpha\lambda^2}{\alpha+\beta\lambda^2}} - \lambda^{-1} & 1 + \sqrt{\frac{\beta+\alpha\lambda^2}{\alpha+\beta\lambda^2}} \end{pmatrix}. \tag{4.4.4}$$

Evidently $\mathcal{M}^* = \mathcal{M}^{-1}$ so that this is actually the decomposition according to Theorem 4.3. The domain of $\mathcal{M}$ is the genus one hyperelliptic curve $\mu^2 + \det D = 0$, the *spectral curve* of the underlying Delaunay surface.

4.5 Finite Blaschke Products. The function f_q occurring in Theorem 4.3 is not always as easy to deal with as in the above example nor can it generally be explicitly computed. Nonetheless, even the trivial case $f_q \equiv 0$ is quite interesting and the corresponding dressing matrices are of such a simple form that the involved Iwasawa decomposition can be explicitly computed.

Lemma: Let $h \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ and F_c be the unitary frame of the round cylinder with monodromy $\chi_c(\tau_q) = \exp((q\lambda^{-1} - \bar{q}\lambda)A)$ for some $q \in \mathbb{C}^*$ with

$$\sqrt{\bar{q}} - \sqrt{q} \notin \pi\,\mathbb{Z}. \tag{4.5.1}$$

Assume $F = h\#F_c$ has a monodromy $h\,\chi_c(\tau_q)\,\exp(f_q\,A)\,h^{-1} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ as in equation (4.3.3). Then the following are equivalent

(i) $h\chi_c(\tau_q)h^{-1} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$

(ii) $f_q \equiv 0$

(iii) $h\,A\,h^{-1}$ is rational on $\mathbb{CP}^1$ with only simple poles.

Proof: We first show that (i) implies (ii). The assumption (i) allows us to write the monodromy of F as $h\,\chi_c(\tau_q)\,h^{-1}\cdot h\exp(f_q\,A)\,h^{-1}$, where the first group of factors is unitary and the second group of factors is contained in $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ and has first term I. Therefore the condition that the whole expression is unitary is equivalent to $h\exp(f_q\,A)\,h^{-1}$ being unitary. But it is also in $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Therefore it is in $\mathbf{U}(1)$. But since it starts with I, it is identically equal to I. As a consequence $\exp(f_q\,A)=I$. Hence $f_q\equiv 2\pi k$ for some $k\in\mathbb{Z}$ and $k=0$, since f_q is an odd function by Lemma 4.3. This shows that (i) implies (ii).

Let us prove that (ii) implies (iii). Recall from (4.3.6) and (4.3.7) that $h\,\exp(p_q\,A)\,h^{-1}=\alpha\,\mathrm{Id}+\beta\,h\,A\,h^{-1}$. Since $f_q=0$ we have $p_q=\lambda^{-1}q-\lambda\bar q$ and therefore $\beta=\sinh(\lambda^{-1}q-\lambda\bar q)$ is holomorphic on $\mathbb{C}^*$. By assumption, $\beta\,h\,A\,h^{-1}$ is holomorphic on $\mathbb{C}^*$ so that $h\,A\,h^{-1}$ is meromorphic on $\mathbb{C}^*$. Let $\lambda_0\in\mathbb{C}^*$ be a root of $\beta=\sinh(\lambda^{-1}q-\lambda\bar q)$, that is, it lies in the set

$$\mathcal{S}_r^q=\left\{\lambda\in A_r : \lambda^{-1}q-\lambda\,\bar q\in\pi i\mathbb{Z}\right\}. \qquad (4.5.2)$$

Then $\beta'(\lambda_0)=\pm(\lambda_0^{-2}q-\bar q)\neq 0$ if and only if $\lambda_0\neq\pm i\sqrt{q/\bar q}$. Our assumption (4.5.1) ensures that $\pm i\sqrt{q/\bar q}\notin\mathcal{S}_r^q$. Hence all roots of β in $\mathbb{C}^*$ are simple. Therefore the entries of $h\,A\,h^{-1}$ can only have simple poles in $\mathbb{C}^*$ which must lie in $\mathcal{S}_r^q$. Note also that $h\,A\,h^{-1}$ is holomorphic at $\lambda=0$. Hence $h\,A\,h^{-1}$ is a holomorphic germ at the origin with a meromorphic extension to $\mathbb{C}^*$. In Section 3.5 of [6] it is shown that the squares of the entries of $h\,A\,h^{-1}$ are finite at ∞. Hence $h\,A\,h^{-1}$ is rational on $\mathbb{CP}^1$ and proves that (ii) implies (iii).

Finally we show that (iii) implies (i). By equations (4.3.6) and (4.3.7) we write $\chi=\alpha\,\mathrm{Id}+\beta\,hAh^{-1}$ and use the fact that $\beta\,h\,A\,h^{-1}$ is holomorphic on $\mathbb{C}^*$. With assumption (iii) this implies that β is meromorphic on $\mathbb{C}^*$. Since α is holomorphic on $\mathbb{C}^*$, so is α^2 and consequently $\beta^2=\alpha^2-1$ is holomorphic on $\mathbb{C}^*$, therefore β itself is holomorphic on $\mathbb{C}^*$. Further, from $\chi^*=\chi^{-1}=\alpha\,\mathrm{Id}-\beta\,hAh^{-1}$ we deduce that α is real on S^1, i.e. $\alpha^*=\alpha$. And since β^2 is real and non-positive on S^1, and since β is holomorphic on $\mathbb{C}^*$, we obtain $\beta^*=-\beta$. Consequently, $(hAh^{-1})^*=hAh^{-1}$ and thus $h\,\chi_c(\tau_q)\,h^{-1}=\exp((\lambda^{-1}q-\lambda\,q)h\,A\,h^{-1})\in\Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$. This proves that (iii) implies (i) and concludes the proof of the lemma. $\square$

Remark: 1. The result above seems to indicate that for every "type of f_q " one obtains an associated type of dressing matrix.

2. In the case $f_q=0$ and the setting of [6], the hyperelliptic surface associated with the dressing is simply the Riemann sphere S^2. If this class would contain some torus, then one would have an example for the "singular tori question" of [1].

3. The set $\mathcal{S}_r^q$ in (4.5.2) is finite for fixed r. Further $\mathcal{S}_r^q\subset\mathbb{R}$ if and only if $q\in i\mathbb{R}$, in which case q also satisfies (4.5.1). Finally, it is easy to see that if $\lambda\in\mathcal{S}_r^q$ then also $\bar\lambda$, $1/\lambda$, $1/\bar\lambda\in\mathcal{S}_r^q$, and in particular, $(\mathcal{S}_r^q)^*=\mathcal{S}_r^q$.

We apply Lemma 4.5 to a class of diagonal dressing matrices for which $f_q\equiv 0$.

Corollary: Let $h=\mathrm{diag}[a,\,1/a]\in\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$. Then h has an extension to $A_{r'}$

for some $r' \in [r,\, 1]$ such that $h^* = h^{-1}$ on $A_{r'}$ and $h\chi_c(\tau_q)h^{-1} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ if and only if a^2 is a finite Blascke product

$$a^2(\lambda) = \prod_{j=1}^{N} \frac{\alpha_j^2 - \lambda^2}{1 - \bar{\alpha}_j^2 \lambda^2},\ \alpha_j \in \mathcal{S}_r^q. \tag{4.5.3}$$

Proof: Assume h has a unitary branch on S^1 and $h\chi_c(\tau_q)h^{-1} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$. Then by Lemma 4.5 we conclude that $h\,A\,h^{-1} = \mathrm{off}[a^2,\, 1/a^2]$ is rational with simple poles and zeroes located in $\mathcal{S}_r^q$. Hence a^2 is of the form (4.5.3). The converse is proven by direct verification. $\square$

Dressing matrices characterised in the previous proposition are thus of the form

$$h = \prod_{j=1}^{N} \begin{pmatrix} \sqrt{\frac{\alpha_j^2-\lambda^2}{1-\bar{\alpha}_j^2\lambda^2}} & 0 \\ 0 & \sqrt{\frac{1-\bar{\alpha}_j^2\lambda^2}{\alpha_j^2-\lambda^2}} \end{pmatrix},\ \alpha_j \in \mathcal{S}_r^q \tag{4.5.4}$$

and are a special instance of an interesting class to which we now turn our attention.

4.6 Simple factors, untwisted case.

In this section we will deal primarily with untwisted loops. The previous discussion has naturally lead us to a class of special dressing matrices, the so called *simple factors* of Uhlenbeck [28] and discussed in similar context in [27] and [4]. The main feature is that dressing with simple factors is explicit. This construction, due to Terng and Uhlenbeck [27] goes as follows:

We decompose $\mathbb{C}^2 = \mathrm{L} \oplus \mathrm{L}^\perp$ for $\mathrm{L} = \mathbb{C}\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. Then for all $A \in \mathbf{GL}(2,\mathbb{C})$ we have

$$\bar{A}^t\mathrm{L} \perp A^{-1}\mathrm{L}^\perp \tag{4.6.1}$$

The hermitian projection $\pi_\mathrm{L} : \mathbb{C}^2 \to \mathrm{L}$ onto L is given by

$$\pi_\mathrm{L} = \frac{1}{|a|^2 + |b|^2} \begin{pmatrix} |a|^2 & a\bar{b} \\ \bar{a}b & |b|^2 \end{pmatrix}.$$

Let $\alpha \in I_1^* = \{\lambda \in \mathbb{C} : 0 < |\lambda| < 1\}$. Then the map

$$\tau_\alpha(\lambda) = \frac{\alpha - \lambda}{1 - \bar{\alpha}\lambda}$$

is invariant under ϱ. For given L and $\alpha \in I_1$, a *simple factor* [27] is a loop of the form

$$\psi_{\alpha,\mathrm{L}}(\lambda) = \pi_\mathrm{L} + \tau_\alpha(\lambda)\,\pi_\mathrm{L}^\perp. \tag{4.6.2}$$

By construction, $\psi_{\alpha,\mathrm{L}} : \mathbb{CP}^1 \setminus \{\alpha,\, 1/\bar{\alpha}\} \to \mathbf{GL}(2,\mathbb{C})$ is analytic and since $|\alpha| > 0$,

$$\psi_{\alpha,\mathrm{L}} \in \Lambda_r^+\mathbf{GL}(2,\mathbb{C}) \text{ for } r < |\alpha|. \tag{4.6.3}$$

Clearly, simple factors are not twisted. Further, since $\psi^*_{\alpha,\mathrm{L}} = \psi^{-1}_{\alpha,\mathrm{L}} = \pi_\mathrm{L} + \tau_\alpha^{-1}\pi_\mathrm{L}^\perp$, we have that

$$\psi_{\alpha,\mathrm{L}} \in \Lambda_r\mathbf{U}(2) \text{ for } r > |\alpha|. \tag{4.6.4}$$

For later use we also note the fact that for any $A \in \mathbf{U}(2)$ we have

$$\psi_{\alpha,A\mathrm{L}} = A\,\psi_{\alpha,\mathrm{L}}\,A^{-1}. \tag{4.6.5}$$

Let $U \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)$, $\alpha \in I_1^*$ and $\mathrm{L} \in \mathbb{CP}^2$ and $\psi_{\alpha,\mathrm{L}}$ be the corresponding simple factor. Then it can be shown, see e.g [17], that

$$\psi_{\alpha,\mathrm{L}}\,U\,\psi^{-1}_{\alpha,\overline{U(\alpha)}^t\mathrm{L}} \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2) \tag{4.6.6}$$

Consequently, we have an explicit r–Iwasawa decomposition of $\psi_{\alpha,\mathrm{L}}\,U$ for $r < |\alpha|$, given by

$$\psi_{\alpha,\mathrm{L}}\,U = \left(\psi_{\alpha,\mathrm{L}}\,U\,\psi^{-1}_{\alpha,\overline{U(\alpha)}^t\mathrm{L}}\right)\psi_{\alpha,\overline{U(\alpha)}^t\mathrm{L}}. \tag{4.6.7}$$

For the geometric applications considered in this paper it is of great value to have a simple and explicit formula for the dressed frame. Simple factors, for which L depends on λ and are of the form

$$\psi_{\alpha,\mathrm{L}(\lambda)} = \pi_{\mathrm{L}(\lambda)} + \tau_\alpha(\lambda)\,(\mathrm{Id} - \pi_{\mathrm{L}(\lambda)}) \tag{4.6.8}$$

will be called *generalised simple factors*. We would like to present an analogous result to 4.6.7 for such generalised simple factors:

Theorem: Let $\mathrm{L} : \mathbb{C}^* \to \mathbb{CP}^1$ be holomorphic and $\langle \mathrm{L}, \mathrm{L}^*\rangle \neq 0$ for all $\lambda \in \mathbb{C}^*$. Let $\alpha \in I_1^*$ and $\psi_{\alpha,\mathrm{L}(\lambda)}$ be the corresponding generalised simple factor. Then for any $U \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)$

$$\psi_{\alpha,\mathrm{L}(\lambda)} \# U = \psi_{\alpha,\mathrm{L}(\lambda)}\,U\,\psi^{-1}_{\alpha,\overline{U(\alpha)}^t\mathrm{L}(\alpha)}. \tag{4.6.9}$$

<u>Proof:</u> Fix $\lambda_0 \in \mathbb{C}^*$ and write $\mathrm{L}(\lambda) = [a(\lambda) : b(\lambda)]$. Then for $\mathrm{L}_0 = [1 : 0]$ and the holomorphic map

$$W(\lambda) = \begin{pmatrix} a(\lambda) & -\overline{b(1/\bar\lambda)} \\ b(\lambda) & \overline{a(1/\bar\lambda)} \end{pmatrix} \tag{4.6.10}$$

we have $\mathrm{L}(\lambda) = W(\lambda)\,\mathrm{L}_0$ and $W(\lambda) \in \Lambda_{\mathbb{C}^*}\mathbf{U}(2)$. Applying 4.6.5 to $A = W(\lambda)$, $\lambda \in \mathbb{C}^*$, we obtain $\psi_{\alpha,\mathrm{L}(\lambda)} = W(\lambda)\,\psi_{\alpha,\mathrm{L}_0}\,W(\lambda)^{-1}$ and

$$\begin{aligned}\psi_{\alpha,\mathrm{L}(\lambda)}\,U(\lambda) &= W(\lambda)\psi_{\alpha,\mathrm{L}_0}W(\lambda)^{-1}U(\lambda) \\ &= \left(W(\lambda)\,\psi_{\alpha,\mathrm{L}_0}\,W(\lambda)^{-1}U(\lambda)\,\psi^{-1}_{\alpha,\mathrm{L}_0'}\right)\psi_{\alpha,\mathrm{L}_0'},\end{aligned}$$

where $\mathrm{L}_0' = \overline{(W(\alpha)^{-1}U(\alpha))}^t\mathrm{L}_0$. The right side is an Iwasawa decomposition. The unitary part can be rewritten in the form

$$W(\lambda)\,\psi_{\alpha,L_0}\,W(\lambda)^{-1}\,U(\lambda)\,\psi^{-1}_{\alpha,L_0'} = \psi_{\alpha,L(\lambda)}\,U(\lambda)\,\psi^{-1}_{\alpha,L_0'}.$$

It is straightforward to verify $\mathrm{L}_0' = \overline{U(\alpha)}^t \mathrm{L}(\alpha)$, thus proving (4.6.9). $\square$

4.7 Twisting of simple factors

Since generalised simple factors are in general untwisted, we need to modify the concept so that it can also be used for the twisted case. We discuss two approaches to deal with simple factors in the twisted case by firstly, taking a simple factor and applying the 'twisting map' and secondly, looking for products of simple factors which happen to be twisted. Applying the 'twisting map' (1.3.1) to a simple factor we obtain

$$\psi_{\alpha,\mathrm{L}(\lambda)} = D(\lambda)\, \psi_{\alpha,\mathrm{L}(\lambda^2)}\, D(\lambda)^{-1} = \psi_{\alpha,D(\lambda)\mathrm{L}(\lambda^2)}.$$

Lemma: a) Twisting a generalised simple factor produces again a generalised simple factor.

b) Let ψ be a simple factor (L is independent of λ). Then the corresponding twisted simple factor is again independent of λ if and only if

$$D\, \pi_{\mathrm{L}}\, D^{-1} = \pi_{\mathrm{L}} \text{ or } \pi_{\mathrm{L}}^{\perp}. \tag{4.7.1}$$

$\square$

Remark: Since $D = D(\lambda)$ is diagonal, the only projections satisfying the condition (4.7.1) above are those onto the canonical basis vectors.

4.8 Twisted products of simple factors As mentioned earlier we also consider products of two simple factors and determine, when such matrices are twisted. Recall the two involutions ϱ, σ defined in (1.1.1) respectively (1.1.2).

Proposition:

a) Let $g \in \Lambda_r\mathbf{SL}(2,\mathbb{C})$. Then $(\sigma g)g$ is twisted if and only if $[\,\sigma g,\, g\,] = 0$.

b) If $g \in \Lambda_r\mathbf{SU}(2)_\sigma$, then $(\sigma g)g \in \Lambda_r\mathbf{SU}(2)_\sigma$.

c) Let $\psi_{\alpha,\mathrm{L}}$ and $\psi_{\beta,\widehat{\mathrm{L}}}$ be simple factors. If $\sigma(\psi_{\alpha,\mathrm{L}})\, \psi_{\beta,\widehat{\mathrm{L}}}$ is twisted, then $\psi_{\alpha,\mathrm{L}} = \psi_{\beta,\widehat{\mathrm{L}}}$.

d) If $\sigma(\psi_{\alpha,\mathrm{L}})\psi_{\alpha,\mathrm{L}}$ is twisted, then either $\sigma(\pi_{\mathrm{L}}) = \pi_{\mathrm{L}}$ or $\sigma(\pi_{\mathrm{L}}) = \pi_{\mathrm{L}}^{\perp}$.

Proof: a) The loop $(\sigma g)g$ is twisted if and only if $\sigma\left((\sigma g)\, g\right) = g\, \sigma g = (\sigma g)\, g$, which is equivalent to $[\sigma g,\, g] = 0$.

b) The loop $g \in \Lambda_r\mathbf{SU}(2)_\sigma$ if and only if $\varrho\, g = g$. Then $\varrho\left((\sigma g)g\right) = (\varrho\sigma g)\,(\varrho g) = (\sigma\, \varrho\, g)\,(\varrho\, g) = (\sigma g)\, g$, since $[\varrho,\, \sigma] = 0$. Hence $(\sigma g)\, g \in \Lambda_r\mathbf{SU}(2)_\sigma$.

c) The product is twisted if and only if $\sigma(\psi_{\alpha,L})\cdot\psi_{\beta,\widehat{L}} = \psi_{\alpha,L}\cdot\sigma(\psi_{\beta,\widehat{L}})$. Let us abbreviate $A = \psi_{\alpha,\mathrm{L}}$ and $B = \psi_{\beta,\widehat{\mathrm{L}}}$. Then the pole of A is at $1/\overline{\alpha}$, while the pole of B is at $1/\overline{\beta}$. On the other hand, the pole of $\sigma(A)$ is at $-1/\overline{\alpha}$ and the analogous result holds for B. Thus comparing the two sides we obtain $\alpha = \beta$. Expanding near a pole and comparing the factors we derive $\mathrm{L} = \widehat{\mathrm{L}}$.

(d) If $(\sigma\psi_{\alpha,\mathrm{L}})\psi_{\alpha,\mathrm{L}}$ is twisted, then by part (a), $[\sigma\psi_{\alpha,\mathrm{L}},\, \psi_{\alpha,\mathrm{L}}] = 0$ which is equivalent to the eigenspaces of $\sigma\psi_{\alpha,\mathrm{L}}$ and $\psi_{\alpha,\mathrm{L}}$ coinciding. Thus there are two possibilities:

$$\sigma_3\, \pi_{\mathrm{L}}\, \sigma_3^{-1} = \pi_{\mathrm{L}} \text{ or } \pi_{\mathrm{L}}^{\perp}. \tag{4.8.1}$$

This proves (d) and concludes the proof of the proposition. □

We are going to evaluate the two possibilities in (4.8.1).

(i) Let us turn to the first case in (4.8.1): For the line $\mathrm{L} = \mathbb{C}\binom{1}{0}$ we have $\psi_{\alpha,\mathrm{L}} = \mathrm{diag}[1,\, \tau_\alpha]$. Using $\tau_\alpha(\lambda)\tau_\alpha(-\lambda) = \tau_{\alpha^2}(\lambda^2)$ we obtain $(\sigma\psi_{\alpha,\mathrm{L}})\psi_{\alpha,\mathrm{L}} = \pi_{\mathrm{L}} + \tau_{\alpha^2}(\lambda^2)\pi_{\mathrm{L}}^{\perp}$. Dividing by the square root of the determinant we arrive at all diagonal twisted simple factors

$$g_{\alpha,\mathrm{L}}(\lambda) = \sqrt{\tau_{\alpha^2}^{-1}(\lambda^2)}\, \pi_{\mathrm{L}} + \sqrt{\tau_{\alpha^2}(\lambda^2)}\, \pi_{\mathrm{L}}^{\perp}. \tag{4.8.2}$$

Then $g_{\alpha,\mathrm{L}}(\lambda) \in \Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ for $0 < r < |\alpha|$ and in matrix form is given by

$$g_{\alpha,\mathrm{L}}(\lambda) = \begin{pmatrix} \sqrt{\frac{1-\bar{\alpha}^2\lambda^2}{\alpha^2-\lambda^2}} & 0 \\ 0 & \sqrt{\frac{\alpha^2-\lambda^2}{1-\bar{\alpha}^2\lambda^2}} \end{pmatrix} \tag{4.8.3}$$

Notice that $g_{\alpha,\mathrm{L}}(\lambda)^{-1}$ corresponds to a factor of the matrix given in (4.5.4).

(ii) Let us turn to the second case in (4.8.1): $\sigma_3\pi_{\mathrm{L}}\sigma_3^{-1} = \pi_{\mathrm{L}}^{\perp}$. For $\mathrm{L} = \mathbb{C}(a,\, b)^t$ and hermitian projection $\pi_{\mathrm{L}} : \mathbb{C}^2 \to \mathrm{L}$ we have $\sigma\pi_{\mathrm{L}} = \pi_{\mathrm{L}}^{\perp}$ if and only if $|a|^2 = |b|^2 = 1/2$. Setting $a = \frac{1}{\sqrt{2}}\exp(is)$ and $b = \frac{1}{\sqrt{2}}\exp(it)$ and rescaling, we may assume without loss of generality that $\mathrm{L} = \mathbb{C}\big(e^{i\theta},\, 1\big)^t$. Then $\pi_{\mathrm{L}} = \frac{1}{2}(\mathrm{Id} + \mathrm{off}[e^{i\theta},\, e^{-i\theta}])$ and for $\psi_{\alpha,\mathrm{L}} = \pi_{\mathrm{L}} + \tau_\alpha(\lambda)\, \pi_{\mathrm{L}}^{\perp}$ we have $\sigma\psi_{\alpha,\mathrm{L}} = \pi_{\mathrm{L}}^{\perp} + \tau_\alpha(-\lambda)\, \pi_{\mathrm{L}}$ and consequently

$$\begin{aligned}(\sigma\psi_{\alpha,\mathrm{L}})\psi_{\alpha,\mathrm{L}} &= \tau_\alpha(\lambda)\, \pi_{\mathrm{L}}^{\perp} + \tau_\alpha(-\lambda)\, \pi_{\mathrm{L}} \\ &= \tfrac{1}{1-\bar{\alpha}^2\lambda^2}((\alpha - \bar{\alpha}\lambda^2)\,\mathrm{Id} + \lambda(1-|\alpha|^2)\,\mathrm{off}[e^{i\theta},\, e^{-i\theta}]).\end{aligned}$$

Normalising, we arrive at the second class of twisted simple factors in $\Lambda_r^+\mathbf{SL}(2,\mathbb{C})_\sigma$ for $0 < r < |\alpha|$, given by

$$g_{\alpha,\mathrm{L}}(\lambda) = \sqrt{\tau_\alpha^{-1}(\lambda)\, \tau_\alpha(-\lambda)}\, \pi_{\mathrm{L}} + \sqrt{\tau_\alpha(\lambda)\, \tau_\alpha^{-1}(-\lambda)}\, \pi_{\mathrm{L}}^{\perp}. \tag{4.8.4}$$

In matrix form, these simple factors look like

$$g_{\alpha,\mathrm{L}}(\lambda) = \sqrt{\frac{1-\bar{\alpha}^2\lambda^2}{\alpha^2-\lambda^2}} \begin{pmatrix} \alpha - \bar{\alpha}\lambda^2 & \lambda(1-|\alpha|^2)e^{i\theta} \\ \lambda(1-|\alpha|^2)e^{-i\theta} & \alpha - \bar{\alpha}\lambda^2 \end{pmatrix}. \tag{4.8.5}$$

The simple factors $g_{\alpha,\mathrm{L}}$ derived in (4.8.2) and (4.8.4) are uniquely determined by their 'singularity' $\alpha \in I_1$ and choice of line $\mathrm{L} \in \mathbb{CP}^1$. Note also that in both cases (4.8.2) and (4.8.4) we have that

$$g_{\alpha,\mathrm{L}} \in \Lambda_r\mathbf{SU}(2)_\sigma \text{ for } |\alpha| < r \leq 1. \tag{4.8.6}$$

The key aspect is that dressing with simple factors is explicit. This idea is due to Terng and Uhlenbeck [27] and has lead to variants as in [4] and [17]. The version we need is proven by twisting Theorem 1.2 in [17] and is as follows

Theorem: Let M be a connected Riemann surface with universal cover $\widetilde{M}$ and let $F(z,\lambda) \in \mathcal{F}(\widetilde{M})$. Let $g_{\alpha,\mathrm{L}} \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ be a simple factor. Then

$$g_{\alpha,\mathrm{L}} \# F = g_{\alpha,\mathrm{L}}\, F\, g_{\alpha,\mathrm{L}'}^{-1} \tag{4.8.7}$$

where $\mathrm{L}' = \overline{F(z,\alpha)}^t \mathrm{L}$ and $g_{\alpha,\mathrm{L}'}^{-1}$ is, pointwise in $z \in \widetilde{M}$, a simple factor of the same form as $g_{\alpha,\mathrm{L}}$. $\square$

We use this factorisation theorem to characterise when it is possible to dress an extended unitary frame with trivial isotropy by a simple factor while retaining a symmetry.

Corollary: Let $\chi(\tau) \in \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$ be a monodromy of an extended unitary frame $F \in \mathcal{F}(\widetilde{M})$. Let $g_{\alpha,\mathrm{L}} \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ be a simple factor. Then $g_{\alpha,\mathrm{L}}\, \chi(\tau)\, g_{\alpha,\mathrm{L}}^{-1}$ is a monodromy of $g_{\alpha,\mathrm{L}} \# F$ with respect to τ if and only if

$$\overline{\chi(\alpha,\tau)}^t \mathrm{L} = \mathrm{L}. \tag{4.8.8}$$

<u>Proof:</u> Let $\hat{F} = g_{\alpha,\mathrm{L}} \# F$. Then $\hat{F} = g_{\alpha,\mathrm{L}}\, F\, g_{\alpha,\mathrm{L}'}^{-1}$ with $\mathrm{L}' = \overline{F(z,\alpha)}^t \mathrm{L}$ by (4.8.7). Let $\tau^* F = \chi(\tau)\, F\, k$. Then

$$\begin{aligned}
\tau^* \hat{F} &= g_{\alpha,\mathrm{L}}\, (\tau^* F)\, g_{\alpha,\tau^*\mathrm{L}'}^{-1} \\
&= g_{\alpha,\mathrm{L}}\, \chi(\tau)\, F\, k\, g_{\alpha,\mathrm{L}''}^{-1} \text{ with } \mathrm{L}'' = \bar{k}^t\, \overline{F(\alpha)}^t\, \overline{\chi(\alpha,\tau)}^t \mathrm{L} = \tau^* \mathrm{L}' \\
&= g_{\alpha,\mathrm{L}}\, \chi(\tau)\, g_{\alpha,\mathrm{L}}^{-1}\, g_{\alpha,\mathrm{L}}\, F\, g_{\alpha,\mathrm{L}_1}^{-1} k \text{ by (4.6.5)}
\end{aligned}$$

where $\mathrm{L}_1 = \overline{F(z,\alpha)}^t\, \overline{\chi(\alpha,\tau)}^t \mathrm{L}$.

If $g_{\alpha,\mathrm{L}}\, \chi(\tau)\, g_{\alpha,\mathrm{L}}^{-1}$ is a monodromy of $\hat{F}$ and we write $\tau^* \hat{F} = g_{\alpha,\mathrm{L}}\, \chi(\tau)\, g_{\alpha,\mathrm{L}}^{-1}\, \hat{F}\, \hat{k}$ and combine with the above, we obtain $g_{\alpha,\mathrm{L}_1}^{-1}\, k = g_{\alpha,\mathrm{L}'}^{-1}\, \hat{k}$. Hence $g_{\alpha,\mathrm{L}_1} g_{\alpha,\mathrm{L}'}^{-1}$ is independent of λ and since there exists a $\lambda_0 \in \mathbb{CP}^1$ at which $\tau_\alpha(\lambda_0) = 1$ we obtain $g_{\alpha,\mathrm{L}_1} g_{\alpha,\mathrm{L}'}^{-1} = \mathrm{Id}$. This implies that $L' = L_1$ which implies (4.8.8). The converse is proven by direct verification. $\square$

5 Dressing non–finite type surfaces

In this section we discuss a family of CMC cylinders found in [16] that arise as perturbations of Delaunay surfaces. The resulting surfaces may possess an arbitrary number of umbilics and are thus not of finite type and are CMC cylinders with one

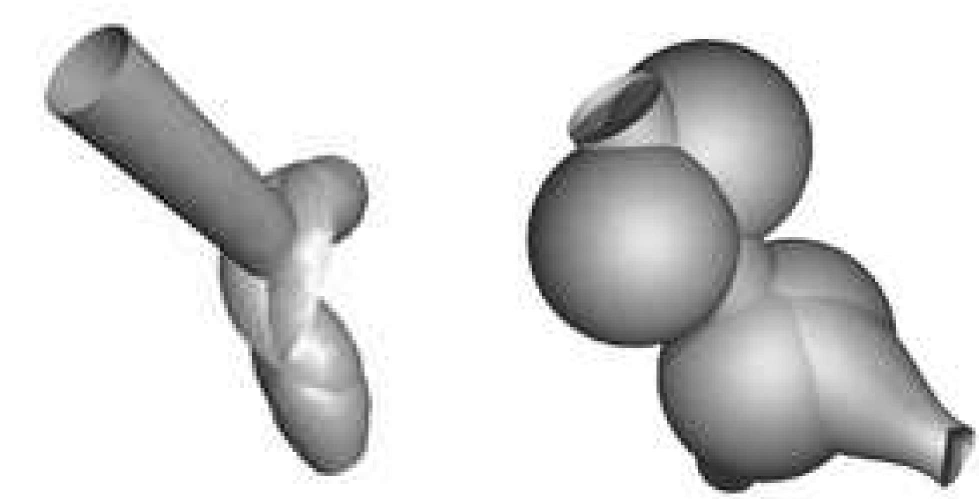

Figure 1: A perturbed round cylinder on the left. A perturbed and dressed round cylinder on the right. Images generated with *CMCLab* [23]. For more images see [24].

Delaunay end [15], see Figure 1. We can dress this class of cylinders with simple factors, having the effect of adding bubbles to the surface, see Figure 2.

5.1 Dressing cylinders with umbilics. We modify a standard result from the theory of differential equations with regular singular points. Since the eigenvalues are λ–dependent, we can avoid the assumption that two elements in the spectrum not differ by an integer by working on an appropriate λ–circle C_r.

Theorem: Let $U_0^* \subset \mathbb{C}$ be an open neighbourhood of $z = 0$ and $\xi \in \Lambda\Omega(U_0^*)$ with a simple pole at $z = 0$ and residue $\mathrm{res}_0\xi = D$ where D has the form (4.4.1). Moreover, we assume that D satisfies the first closing condition (5.1.1). Then there exists an $r \in (0, 1)$ and a solution of $d\Psi = \Psi\xi$ of the form $\Psi = z^D P$ with $P : U_0 \to \Lambda_r\mathbf{SL}(2, \mathbb{C})_\sigma$ holomorphic. Further, Ψ has Delaunay monodromy and is the holomorphic frame associated with $(\xi,\, P(1),\, 1)$.

Proof: Let $U_0 = U_0^* \cup \{0\}$ and write $\xi = D\frac{dz}{z} + \eta$ with $\eta \in \Lambda\Omega_{U_0}$. The differential equation for P is $dP = P\eta + [P,\, D\frac{dz}{z}]$. We will show that this differential equation has a holomorphic solution. Expanding $P = \sum_{k=0}^{\infty} P_k z^k$ and $\xi = (\frac{1}{z}D + \Sigma\eta_k z^k)dz$ at $z = 0$, the coefficients are recursively given by

$$kP_k + [D, P_k] = \sum_{r+s=k-1} P_r\eta_s.$$

By our Ansatz $P_k = 0$ for $k < 0$ and we are free to choose $P_0 \in \Lambda_r\mathbf{SL}(2, \mathbb{C})_\sigma$ with $[D, P_0] = 0$. (Note this freedom only means $P_0 = \alpha\mathrm{Id} + \beta D$, since D is a semisimple $2 \times 2-$matrix with different eigenvalues.) The eigenvalues of D, are of the form $\pm\mu$, where μ is a solution to the equation $\mu^2 = -\det D$. Then the operators $k\mathrm{Id} + \mathrm{ad_D}$, $k \in \mathbb{Z}$, have spectrum $\sigma = \{k,\, k \pm 2\mu\}$. We are only interested in $k \geq 1$, in which case $0 \notin \sigma$ if and only if $\mu \notin \frac{1}{2}\mathbb{N}$.

We need to show that there exists an $r \in (0,\,1)$ for which $\mu(\lambda) \notin \frac{1}{2}\mathbb{N}$ for every $\lambda \in C_r$. A priori, $\mu(\lambda) \in \mathbb{R}$ if and only if either $\lambda \in S^1$ or $\lambda \in \mathbb{R}$. Since we seek $r \in (0,\,1)$ we have $\mu(\lambda) \in \mathbb{R}$ if and only if $\lambda \in \mathbb{R}$ we need to ensure $\mu(\pm r) \notin \frac{1}{2}\mathbb{N}$. Since D satisfies the first closing condition by assumption, we know that $\mu(1) = n/2$ for some $n \in \mathbb{Z}$ the winding number of the Delaunay surface. A direct computation shows $\mu(\pm r) \notin \frac{1}{2}\mathbb{Z}$ if and only if $r^{-2} + r^2 \neq \pm(k^2 - n^2)/(4ab) + 2$ for all $k \in \mathbb{N}$. The sequences $a_k^{\pm} := \pm(k^2 - n^2)/(4ab) + 2$ are monotonic, hence we can always find an $0 < r < 1$ for which $r^{-2} + r^2 \notin \{a_k^{\pm}\}$.

This proves the existence of a formal power series solving the differential equation for P. The coefficients of this formal power series are defined for all $z \in U_0$. The domain of analyticity of P is U_0 by standard ODE arguments, see e.g [12]. It is shown in [11] that P has the twisted λ behaviour and that the coefficients of P as functions of λ are in $\mathcal{A}_r$. To show that Ψ has Delaunay monodromy, recall from 4.4 that the Delaunay monodromy is $\exp(2\pi i D)$. For the translation $\tau(z) = z + 2\pi i$ the monodromy of Ψ is $\tau^*\Psi\Psi^{-1} = \tau^* z^D \tau^* P P^{-1} z^{-D} = \exp(2\pi i D)$. □

For our purposes, the specific form of Ψ proven above is of great importance to us, since it gives easy control over the monodromy singled out by the singularity at D. In particular, if we dress the corresponding frame by some simple factor it is easy to determine the dressed monodromy, at least if the associated surface has umbilical points. From a geometric point of view this is particularly interesting, since by a result of [19], the classical Bianchi–Bäcklund transformation does correspond to dressing by some simple factor. Actually, in the twisted setup, our 'simple factors' are in fact a product of two simple factors. These correspond to the two step procedure of the Bianchi–Bäcklund transformation. As a matter of fact, by making the right choice for the singularity of the simple factor, is is possible to control the topology of the resulting surface.

Remark: 1) The proof of the Theorem above actually shows that there is some $0 < r < 1$ such that for all $0 < r \leq r' < 1$ the claim holds.

2) Instead of choosing some $0 < r < 1$, one could ask for what potentials ξ one can obtain a solution of the form $\Psi = z^D P$ without singularities on S^1. Once this is achieved, one can conclude as above and infer statements about the monodromy and thus one can clear the way for some understanding for the effect of dressing by simple factors. Such a condition has been given in [11].

In order to dress a CMC surface with non-trivial topology into a new CMC surface with the same topology, we recall from [6] when for a given triple $(\xi, \Phi_0, \tilde{z}_0)$ one specific member of the associated family is invariant under Δ, thus characterising the period problem in the DPW framework:

Let f_λ be an associated family. Let F be the extended unitary frame of the surface with monodromy $\chi : \Delta \to \Lambda_{\mathbb{C}^*}\mathbf{SU}(2)_\sigma$. Then there exists a $\lambda_0 \in S^1$ such that

$\tau^* f_{\lambda_0} = f_{\lambda_0}$ for all $\tau \in \Delta$ if and only if for all $\tau \in \Delta$, χ satisfies both

$$\chi(\tau,\, \lambda_0) = \pm \mathrm{Id}, \tag{5.1.1}$$

$$\tfrac{\partial}{\partial \lambda}\big|_{\lambda_0} \chi(\tau,\, \lambda) = 0. \tag{5.1.2}$$

Corollary: Let $\xi \in \Lambda\Omega(\mathbb{C}^*)$ as in Theorem 5.1 and $g_{\alpha,\mathrm{L}} \in \Lambda_r^+ \mathbf{SL}(2,\mathbb{C})_\sigma$ be a simple factor. Choosing $r \in (0,\, 1)$ as above and $\alpha \in \mathbb{C}$ with $|\alpha| \in (r,\, 1)$ such that

$$\det D(\alpha) = \tfrac{n}{4} \text{ for some } n \in \mathbb{Z}, \tag{5.1.3}$$

then $(\xi,\, g_{\alpha,\mathrm{L}}\, P(1),\, 1)$ generates a CMC cylinder (with winding number n).

Proof: Let $\Phi = z^D P$ and F be the holomorphic respectively unitary frame generated by $(\xi,\, P(1),\, 1)$. Both have Delaunay monodromy $\chi(\tau,\, \lambda) = \exp(2\pi i D(\lambda))$ with respect to the translation $\tau(z) = z + 2\pi i$. The condition (5.1.3) ensures that $\chi(\tau,\, \alpha) = \pm\mathrm{Id}$ and consequently (4.8.8) holds for any choice of line $\mathrm{L} \in \mathbb{CP}^1$. Hence, by Corollary 4.8, $\hat{\chi}(\tau) = g_{\alpha,\mathrm{L}}\, \chi(\tau)\, g_{\alpha,\mathrm{L}}^{-1}$ is a monodromy of $g_{\alpha,\mathrm{L}} \# F$. The conditions (5.1.1) and (5.1.2) are verified using the facts $\chi(\tau, 1) = \pm\mathrm{Id}$ and $\partial_\lambda \chi(\tau)|_{\lambda=1} = 0$. $\square$

5.2 Dressing 3-Noids. Let us briefly outline how the above theory can be applied to construct the dressed 3-Noids of [17]. Let $\mathcal{T} = \mathbb{CP}^1 \setminus \{0,\, 1,\, \infty\}$ and $f : \mathcal{T} \to \mathbb{R}^3$ a CMC 3-Noid with Delaunay ends as constructed in [11] or [25]. Let $F \in \mathcal{F}(\widetilde{\mathcal{T}})$ be an extended frame such that at $\lambda = 1$ the immersion f is obtained via the Sym–Bobenko formula (1.4.3). Then there are three monodromies, χ_1, χ_2, χ_3, one for each end, which can be computed in terms of Γ–functions [11], [17] and satisfy $\prod_{j=1}^3 \chi_j = \mathrm{Id}$.

In [17] it was shown that there exist values $\alpha \in \mathbb{C}^*$, $|\alpha| < 1$ and invariant subspaces $\mathrm{L} \in \mathbb{CP}^1$ such that (4.8.8) holds for all three χ_j's. Consequently, by Corollary 4.8 the unitary frame obtained by dressing with the simple factor $g_{\alpha,\mathrm{L}}$ has monodromies $\hat{\chi}_j = g_{\alpha,\mathrm{L}}\, \chi_j\, g_{\alpha,\mathrm{L}}^{-1}$ and since the closing conditions (5.1.1) and (5.1.2) are invariant under conjugation, the resulting surface is again a CMC 3–Noid.

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